

CONJUGACY AND COCYCLE CONJUGACY OF AUTOMORPHISMS OF \mathcal{O}_2 ARE NOT BOREL

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ABSTRACT. The group of automorphisms of the Cuntz algebra \mathcal{O}_2 is a Polish group with respect to the topology of pointwise convergence in norm. Our main result is that the relations of conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 are complete analytic sets and, in particular, not Borel. Moreover, we show that from the point of view of Borel complexity theory, classifying automorphisms of \mathcal{O}_2 up to conjugacy or cocycle conjugacy is strictly more difficult than classifying up to isomorphism any class of countable structures with Borel isomorphism relation. In fact the same conclusions hold even if one only considers automorphisms of \mathcal{O}_2 of a fixed finite order. In the course of the proof we will show that the relation of isomorphism of Kirchberg algebras (with trivial K_1 -group and satisfying the Universal Coefficient Theorem) is a complete analytic set. Moreover, it is strictly more difficult to classify Kirchberg algebras (with trivial K_1 -group and satisfying the Universal Coefficient Theorem) than classifying up to isomorphism any class of countable structures with Borel isomorphism relation.

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1. INTRODUCTION

The *Cuntz algebra* \mathcal{O}_2 can be described as the universal unital C^* -algebra generated by two isometries s_1 and s_2 subject to the relation

$$s_1 s_1^* + s_2 s_2^* = 1.$$

It was defined and studied by Cuntz in the groundbreaking paper [10]. Since then, a stream of results has made clear the key role of \mathcal{O}_2 in the classification theory of C^* -algebras; see [51, Chapter 2] for a complete account and more references. This has served as motivation for an intensive study of the structural properties of \mathcal{O}_2 and its automorphism group, as in [5–9, 38, 59]. In particular, a lot of effort has been put into trying to classify several important classes of automorphisms; see for example [26, 27].

If A is a separable C^* -algebra, then the group $\text{Aut}(A)$ of automorphisms of A is a Polish group with respect to the topology of pointwise convergence in norm. Two automorphisms α and β of A are said to be *conjugate* if there exists an automorphism γ of A such that

$$\gamma \circ \alpha \circ \gamma^{-1} = \beta.$$

When A is unital, every unitary u in A defines an automorphism via $a \mapsto uau^*$, and automorphisms of this form are called *inner*. The set $\text{Inn}(A)$ of all inner automorphisms of A is a normal subgroup of $\text{Aut}(A)$, and two automorphisms α and β of A are said to be *cocycle conjugate* if their images in the quotient $\text{Aut}(A)/\text{Inn}(A)$ are conjugate.

Recall that a topological space is said to be *Polish* if it is separable and its topology is induced by a complete metric. A *Polish group* is a topological group whose topology is Polish. A *standard Borel space* is a set endowed with a σ -algebra which is the σ -algebra of Borel sets for some Polish topology on the space. It is not difficult to verify that, under the assumption that A is separable, its automorphism group $\text{Aut}(A)$ is a Polish group with respect to the topology of pointwise convergence in norm.

Definition. A subset B of a standard Borel space X is said to be *analytic* if it is the image of a standard Borel space under a Borel function.

If B and C are analytic subsets of the standard Borel spaces X and Y , then B is said to be *Wadge reducible* to C if there is a Borel map $f: X \rightarrow Y$ such that B is the inverse image of C under f ; see [31, Section 2.E]. An analytic set which is moreover a maximal element in the class of analytic sets under Wadge reducibility is called a *complete analytic set*; more information can be found in [31, Section 26.C].

It is a classical result of Souslin from the early beginnings of descriptive set theory, that there are analytic sets which are not Borel [56]. In particular—since set that is Wadge reducible to a Borel set is Borel—a complete analytic set is not Borel.

The main result of this paper asserts that the relations of conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 are complete analytic sets when regarded as subsets of $\text{Aut}(\mathcal{O}_2) \times \text{Aut}(\mathcal{O}_2)$, and in particular not Borel.

Informally speaking, a set (or function) is Borel whenever it can be computed by a countable protocol whose basic bit of information is membership in open sets. The fact that a set X is not Borel can be interpreted as the assertion that the problem of membership in X can not be decided by such a countable protocol, and it is therefore highly intractable. We can therefore reformulate the main result of this paper as follows: There does not exist any countable protocol able to determine whether a given pair of automorphisms of \mathcal{O}_2 are conjugate or cocycle conjugate by only looking at any given stage of the computation at the value of the given automorphisms at some arbitrarily large finite set of elements of \mathcal{O}_2 up to some arbitrarily small strictly positive error.

The fact that conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 are not Borel should be compared with the fact that for any separable C^* -algebra A , the relation of unitary equivalence of automorphisms of A is Borel. This is because the relation of coset equivalence modulo the Borel subgroup $\text{Inn}(A)$ of $\text{Aut}(A)$. (This does not necessarily mean that the problem of classifying the automorphisms of A up to unitary equivalence is more tractable: It is shown in [37] that whenever A is simple –or just does not have continuous trace– then the automorphisms of A cannot be classified up to unitary equivalence using countable structures as invariants.) Similarly, the spectral theorem for unitary operators on the Hilbert space shows that the relation of conjugacy of unitary operators is Borel; more details can be found in [20, Example 55]. On the other hand, the main result of [20] asserts that the relation of conjugacy for ergodic measure-preserving transformations on the Lebesgue space is also complete analytic.

We will moreover show that classifying automorphisms of \mathcal{O}_2 up to either conjugacy or cocycle conjugacy is strictly more difficult than classifying any class of countable structures with Borel isomorphism relation. This statement can be made precise within the framework of invariant complexity theory. In this context, classification problems are regarded as equivalence relations on standard Borel spaces. Virtually any concrete classification problem in mathematics can be regarded –possibly after a suitable parametrization– as the problem of classifying the elements of some standard Borel space up to some equivalence relation. The key notion of comparison between equivalence relations is the notion of Borel reduction.

Definition. Suppose that E and F are equivalence relation on standard Borel spaces X and Y . A *Borel reduction* from E to F is a Borel function $f: X \rightarrow Y$ such that

$$xE x' \quad \text{if and only if} \quad f(x) F f(x').$$

A Borel reduction from E to F can be regarded as a way to assign –in a constructive way– to the objects of X , equivalence classes of F as complete invariants for E .

Definition. The equivalence relation E is said to be *Borel reducible* to F , in symbol $E \leq_B F$, if there is a Borel reduction from E to F .

In this case, the equivalence relation F can be thought of as being more complicated than E , since any Borel classification of the objects of Y up to F entails –by precomposing with a Borel reduction from E to F – a Borel classification of objects of X up to E . It is immediate to check that if E is Borel reducible to F , then E (as a subset of $X \times X$) is Wadge reducible to F (as a subset of $Y \times Y$). In particular, if E is a complete analytic set and $E \leq_B F$, then F is a complete analytic set. Observe that if F is an equivalence relation on Y , and X is an F -invariant Borel subset of Y , then the restriction of F to X is Borel reducible to F .

Using this terminology, we can reformulate the assertion about the complexity of the relations of conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 as follows. If \mathcal{C} is any class of countable structures such that the corresponding isomorphism relation $\cong_{\mathcal{C}}$ is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to both conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 . Furthermore, if E is any Borel equivalence relation, then the relations of conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 are *not* Borel reducible to E . In particular this rules out any classification that uses as invariant Borel measures on a Polish space (up to measure equivalence) or unitary operators on the Hilbert space (up to conjugacy). In fact, as observed before, the relations of measure equivalence and, by the spectral theorem, the relation of conjugacy of unitary operators are Borel; see [20, Example 55].

All the results mentioned so far about the complexity of the relation of conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 will be shown to hold even if one only considers automorphisms of a fixed finite order. Moreover, it will follow from the argument that the same assertions hold for the relation of isomorphism of Kirchberg algebras (with trivial K_0 -group and satisfying the Universal Coefficient Theorem). It also follows from our constructions and [16, Theorem 1.11] that, for every $n \in \mathbb{N}$, the relation of isomorphisms of unital AF-algebras with K_0 -group of rank $n + 1$ is strictly more complicate than the relation of isomorphism of unital AF-algebras with K_0 -groups of rank n .

It should be mentioned that it is a consequence of the main result of [33] that the automorphisms of \mathcal{O}_2 are not classifiable up to conjugacy by countable structures. This means that there is no explicit way to assign a countable structure to every automorphism of \mathcal{O}_2 , in such a way that two automorphisms are conjugate if and only if the corresponding structures are isomorphic. More precisely, for no class \mathcal{C} of countable structures, is the relation of conjugacy of automorphisms of \mathcal{O}_2 Borel reducible to the relation of isomorphisms of elements of \mathcal{C} . Moreover the same conclusions hold for

any set of automorphisms of \mathcal{O}_2 which is not meager in the topology of pointwise convergence. Similar conclusions hold for automorphisms of any separable C*-algebra absorbing the Jiang-Su algebra tensorially.

The strategy of the proof of the main theorem is as follows. Using techniques from [13, 25], we show that for every prime number p , the relation of isomorphism of countable p -divisible torsion free abelian groups is a complete analytic set, and it is strictly more complicated than the relation of isomorphism of any class of countable structures with Borel isomorphism relation. We then show that the relation of isomorphism of p -divisible abelian groups is Borel reducible to the relations of conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 of order p .

This is achieved by showing that there is a Borel way to assign to a countable abelian group G to assign to a countable abelian group a Kirchberg algebra A_G with trivial K_1 -group, K_0 -group isomorphic to G , and with the class of the unit in K_0 being the zero element. Adapting a construction of Izumi from [26], we define an automorphism ν_p of \mathcal{O}_2 of order p with the following property: Tensoring the identity automorphism of A_G by ν_p , and identifying $A_G \otimes \mathcal{O}_2$ with \mathcal{O}_2 by Kirchberg's absorption theorem, gives a reduction of isomorphism of Kirchberg algebras with p -divisible K_0 -group and with the class of the unit being the trivial element in K_0 , to conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 of order p . The proof is concluded by showing –using results from [19]– that such reduction is implemented by a Borel map.

The present paper is organized as follows. Section 2 presents a functorial version of the notion of standard Borel parametrization of a category as defined in [19]. Several functorial parametrizations for the category are then presented and shown to be equivalent. Finally, many standard constructions in C*-algebra theory are shown to be computable by Borel maps in these parametrizations. The main result of Section 3 asserts that the reduced crossed product of a C*-algebra by an action of a countable discrete group can be computed in a Borel way. The same conclusion holds for crossed products by a corner endomorphism in the sense of [2]. Section 4 provides a Borel version of the correspondence between unital AF-algebras and dimension groups established in [14, 15]. We show that there is a Borel map that assigns to a dimension group D , a unital AF-algebra B_D such that D is isomorphic to the K_0 -group of B_D . Moreover, given an endomorphism β of D , one can select in a Borel fashion an endomorphism $\rho_{D,\beta}$ of B_D whose induced endomorphism of $K_0(B_D)$ is conjugate to β . Finally, Section 5 contains the proof of the main results.

In the following, all C*-algebras and Hilbert spaces are assumed to be *separable*, and all discrete groups are assumed to be *countable*. We denote by ω the set of natural numbers *including* 0. An element $n \in \omega$ will be

identified with the set $\{0, 1, \dots, n-1\}$ of its predecessors. (In particular 0 is identified with the empty set.) We will therefore write $i \in n$ to mean that i is a natural number and $i < n$.

For $n \geq 1$, we write \mathbb{Z}_n for the cyclic group $\mathbb{Z}/n\mathbb{Z}$.

If X is a Polish space and D is a countable set, we endow the set X^D of D -indexed sequences of elements of X with the product topology. Likewise, if X is a standard Borel space, then we give X^D the product Borel structure. In the particular case where $X = 2 = \{0, 1\}$, we identify 2^D with the set of subsets of D with its Cantor set topology, and the corresponding standard Borel structure. In the following we will often make use –without explicit mention– of the following basic principle: Suppose that X is a standard Borel space, D is a countable set, and B is a Borel subset of $X \times D$ such that for every $x \in X$ there is $y \in D$ such that $(x, y) \in B$. Then there is a Borel selector for B , this is, a function f from X to D such that $(x, f(x)) \in B$ for every $x \in X$. To see this one can just fix a well order $<$ of D and define $f(x)$ to be the $<$ -minimum of the set of $y \in D$ such that $(x, y) \in B$.

Moreover we will use throughout the paper the fact that a G_δ subspace of a Polish space is Polish in the subspace topology [31, Theorem 3.11], and that a Borel subspace of a standard Borel space is standard with the inherited Borel structure [31, Proposition 12.1].

We have tried to make this paper accessible to operator algebraists who are not familiar with descriptive set theory, as well as set theorists who are not familiar with C^* -algebras.

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2. PARAMETRIZING THE CATEGORY OF C^* -ALGEBRAS

2.1. Background on C^* -algebras and notation. For a Hilbert space H , we denote by $B(H)$ the algebra of bounded operators on H , and by $\mathcal{K}(H)$ the algebra of compact operators on H . The set of $T \in B(H)$ of operator norm at most 1 is denoted by $B_1(H)$. The weak operator topology on $B(H)$ is the weakest topology making the functions

$$\begin{aligned} B(H) &\mapsto \mathbb{C} \\ x &\mapsto \langle x\xi, \eta \rangle \end{aligned}$$

for $\xi, \eta \in H$ continuous. Recall that addition and scalar multiplication are jointly weakly continuous on $B(H)$, while composition of operators is only separately continuous in each variable; see [1, I.3.2.1]. The unit ball $B_1(H)$ of $B(H)$ is compact when endowed with the weak topology; see [1, I.3.2.4].

We denote by $U(H)$ the group of unitaries in H . It is easily checked that $U(H)$ is a G_δ subset of $B(H)$ with respect to the weak topology. Therefore the weak topology makes $U(H)$ a Polish group by [31, Corollary 9.5]. It is well known that on $U(H)$ the weak topology coincides with several other

operator topologies, such as the weak, strong, σ -weak, and σ -strong operator topology; see [1, I.3.2.9].

A C^* -algebra is a subalgebra of the algebra $B(H)$ of bounded linear operators on a Hilbert space H that is closed in the norm topology and contains the adjoint of any of its elements. In particular $B(H)$ is itself a (nonseparable) C^* -algebra, and $K(H)$ is a separable C^* -algebra.

Equivalently, C^* -algebras can be abstractly characterized as those Banach $*$ -algebras A whose norm satisfies the C^* -identity

$$\|a^*a\| = \|a\|^2$$

for all a in A . A C^* -algebra is *unital* if it contains a multiplicative identity (called *unit*) usually denoted by 1. An ideal of a C^* -algebra A is an ideal of A in the ring-theoretic sense. A C^* -algebra is *simple* if it contains no nontrivial closed ideals [1, II.5.4.1]. If A and B are C^* -algebras, a $*$ -homomorphism from A to B is an algebra homomorphism $\varphi: A \rightarrow B$ satisfying $\varphi(a^*) = \varphi(a)^*$ for all a in A . It is a classical result of the theory of C^* -algebras [1, II.1.6.6] that if $\varphi: A \rightarrow B$ is a $*$ -homomorphism, then φ is contractive, this is, $\|\varphi(a)\| \leq \|a\|$ for every $a \in A$, and moreover φ is isometric if and only if it is injective. As a consequence, the range of any $*$ -homomorphism is automatically closed. A $*$ -isomorphism between A and B is a bijective $*$ -homomorphism. Note that $*$ -isomorphisms are necessarily isometric. A *representation* of a C^* -algebra A on a Hilbert space H is a $*$ -homomorphism from A to $B(H)$. A representation is *faithful* if it is injective or –equivalently– isometric.

An *automorphism* of a C^* -algebra A is a $*$ -isomorphism from A to A . The set of all automorphisms of A , denoted by $\text{Aut}(A)$, is a Polish group under composition when endowed with the topology of pointwise norm convergence. In this topology, a sequence $(\varphi_n)_{n \in \omega}$ in $\text{Aut}(A)$ converges to an automorphism φ if and only if

$$\lim_{n \rightarrow \infty} \|\varphi_n(a) - \varphi(a)\| = 0$$

for every $a \in A$. Two automorphisms of A are said to be *conjugate* if they are conjugate elements of $\text{Aut}(A)$.

If A is unital, an element u of A is said to be a *unitary* if $uu^* = u^*u = 1$. The unitary elements of A form a group under multiplication, denoted by $U(A)$. Any unitary element u of A defines an automorphism $\text{Ad}(u)$ of A , which is given by

$$\text{Ad}(u)(a) = uau^*$$

for all a in A . Automorphisms of this form are called *inner*, and form a normal subgroup $\text{Inn}(A)$ of $\text{Aut}(A)$.

Definition 2.1.1. Let G be a countable discrete group, and let A be a C^* -algebra. An *action* α of G on A is a group homomorphism $g \mapsto \alpha_g$ from G to the group $\text{Aut}(A)$ of automorphisms of A .

Two actions α and β of G on A are said to be *conjugate* if there is $\gamma \in \text{Aut}(A)$ such that

$$\gamma \circ \alpha_g \circ \gamma^{-1} = \beta_g$$

for every $g \in G$.

Let A be a unital C^* -algebra and let α be an action of G on A . An α -cocycle is a function $u: G \rightarrow U(A)$ satisfying

$$u_{gh} = u_g \alpha_g(u_h)$$

for every $g, h \in G$.

If u is an α -cocycle, we define the u -perturbation of α , denoted $\alpha^u: G \rightarrow \text{Aut}(A)$, by

$$\alpha_g^u = \text{Ad}(u_g) \circ \alpha_g$$

for g in G .

Two actions α and β of G on A are said to be *cocycle conjugate* if β is conjugate to a perturbation of α by a cocycle.

It is not hard to check that the relation of cocycle conjugacy is an equivalence relation for actions. In the case when G is the group of integers \mathbb{Z} , actions of \mathbb{Z} on A naturally correspond to single automorphisms of A . Similarly, if G is the group \mathbb{Z}_n , then actions of \mathbb{Z}_n on A correspond to automorphisms of A whose order divides n . We show in Lemma 2.1.2 below that the notions of conjugacy and cocycle conjugacy for actions and automorphisms are respected by this correspondence when A has trivial center. These observations will be used to infer Corollary 5.7.4 from Corollary 5.7.3.

Lemma 2.1.2. *Suppose that α and β are automorphisms of a unital C^* -algebra A .*

- (1) *Then the following statements are equivalent:*
 - (a) *The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of \mathbb{Z} on A are cocycle conjugate;*
 - (b) *There are an automorphism γ of A and a unitary u of A such that $\text{Ad}(u) \circ \alpha = \gamma \circ \beta \circ \gamma^{-1}$.*
- (2) *Assume moreover that α and β have order $k \geq 2$ and that A has trivial center (for example, if A is simple). Then the following statements are equivalent:*
 - (a) *The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of \mathbb{Z}_k on A are cocycle conjugate;*
 - (b) *The actions $n \mapsto \alpha^n$ and $n \mapsto \beta^n$ of \mathbb{Z} on A are cocycle conjugate.*

Proof. (1). To show that (a) implies (b), simply take the unitary $u = u_1$ coming from the α -cocycle $u: \mathbb{Z} \rightarrow U(A)$.

Conversely, if u is a unitary in A as in the statement, we define an α -cocycle as follows. Set $u_0 = 1$ and $u_1 = u$, and for $n \geq 2$ define u_n inductively by $u_n = u_1 \alpha(u_{n-1})$. Set $u_{-1} = \alpha^{-1}(u_1^*)$, and for $n \leq -2$, define u_n inductively by $u_n = u_{-1} \alpha^{-1}(u_{n+1})$. It is straightforward to check that

$n \mapsto u_n$ is an α -cocycle, and that the automorphism γ in the statement implements the conjugacy between α^u and β .

(2). To show that (a) implies (b), it is enough to note that if $u: \mathbb{Z}_k \rightarrow U(A)$ is an α -cocycle, when we regard α as a \mathbb{Z}_k action, then the sequence $(v_m)_{m \in \mathbb{N}}$ of unitaries in A given by $v_m = u_n$ if $m = n \bmod k$, is an α -cocycle, when we regard α as a \mathbb{Z} action.

Assume that α and β are cocycle conjugate as automorphisms of A . Let $(u_n)_{n \in \mathbb{N}}$ be an α -cocycle and let γ be an automorphism implementing the conjugacy. Fix n in \mathbb{N} , and write $n = km + r$ for uniquely determined $k \in \mathbb{Z}$ and $r \in k$. Since α and β have order k , we have

$$\begin{aligned} \text{Ad}(u_{km+r}) \circ \alpha^r &= \text{Ad}(u_{km+r}) \circ \alpha^{km+r} \\ &= \gamma \circ \beta^{km+r} \circ \gamma^{-1} \\ &= \gamma \circ \beta^r \circ \gamma^{-1} \\ &= \text{Ad}(u_r) \circ \alpha^r. \end{aligned}$$

In particular, $\text{Ad}(u_{n+mk}) = \text{Ad}(u_n)$, so u_{n+mk} and u_n differ by a central unitary. Since the center of A is trivial, upon correcting by a scalar, we may assume that $u_{n+mk} = u_n$.

Thus, the assignment $v: \mathbb{Z}_k \rightarrow U(A)$ given by $n \mapsto u_n$ is an α -cocycle, when we regard α as a \mathbb{Z}_k action, and γ implements a conjugacy between the \mathbb{Z}_k actions α^v and β . This finishes the proof. \square

More generally, one can define actions of locally compact groups on C^* -algebras, as well as conjugacy and cocycle conjugacy for such actions. More details can be found in [1, Section II.1].

If A and B are C^* -algebras, the tensor product of A and B as complex algebras with involution is denoted by $A \odot B$ and called *algebraic tensor product* of A and B [1, II.9.1.1]. A C^* -algebra A is *nuclear* or *amenable* if, for any other C^* -algebra B , the algebraic tensor product $A \odot B$ bears a unique C^* -norm, this is, a not necessarily complete norm satisfying

$$\|xy\| \leq \|x\| \|y\|$$

and

$$\|x^*x\| = \|x\|^2.$$

The completion of $A \odot B$ with respect to such norm is called the (C^* -algebra) *tensor product* of A and B and denoted by $A \otimes B$. All the C^* -algebras considered in Section 5 will be nuclear, so their tensor product $A \otimes B$ will be well defined. It is hard to overestimate the importance of nuclearity in the theory of C^* -algebras. Nuclearity is the analog for C^* -algebras of amenability for groups and Banach algebras. Nuclear C^* -algebras admit several equivalent characterizations; see [1, IV.3.1.5, IV.3.1.6, IV.3.1.12]. Moreover they constitute the main focus of Elliott classification program [51, Section 2.2].

2.2. Functorial parametrization. Recall that a *(small) semigroupoid* is a quintuple $(X, \mathcal{C}_X, s, r, \cdot)$, where X and \mathcal{C}_X are sets, s, r are functions from \mathcal{C}_X to X , and \cdot is an associative partially defined binary operation on \mathcal{C}_X with domain

$$\{(x, y) \in \mathcal{C}_X \times \mathcal{C}_X : r(x) = s(y)\}$$

such that $r(x \cdot y) = r(y)$ and $s(x \cdot y) = s(x)$ for all x and y in \mathcal{C}_X . The elements of X are called objects, the elements of \mathcal{C}_X morphisms, the map \cdot composition, and the maps s and r source and range map. In the following, a semigroupoid $(X, \mathcal{C}_X, s, r, \cdot)$ will be denoted simply by \mathcal{C}_X . Note that a (small) category is precisely a (small) semigroupoid, where moreover the identity arrow $\text{id}_x \in \mathcal{C}_X$ is associated with the element x of X . A morphism between semigroupoids \mathcal{C}_X and $\mathcal{C}_{X'}$ is a pair (f, F) of functions $f: X \rightarrow X'$ and $F: \mathcal{C}_X \rightarrow \mathcal{C}_{X'}$ such that

- $s_{X'} \circ F = f \circ s_X$,
- $r_{X'} \circ F = f \circ r_X$, and
- $F(a \cdot b) = F(a) \cdot F(b)$ for every a and $b \in \mathcal{C}_X$.

In the case of categories, a morphism of semigroupoids is just a functor.

A *standard Borel semigroupoid* is a semigroupoid \mathcal{C}_X such that X and \mathcal{C}_X are endowed with standard Borel structures making the source and range functions s and r Borel.

Definition 2.2.1. Let \mathcal{D} be a category, let \mathcal{C}_X be a standard Borel semigroupoid, and let (f, F) be a morphism from \mathcal{C}_X to \mathcal{D} . We say that (\mathcal{C}_X, f, F) is a *good parametrization* of \mathcal{D} if

- (f, F) is *essentially surjective*, this is, if every object of \mathcal{D} is isomorphic to an object in the range of f ,
- (f, F) is *full*, this is, if for every $x, y \in X$ the set $\text{Hom}(f(x), f(y))$ is contained in the range of F , and
- the set Iso_X of elements of \mathcal{C}_X that are mapped by F to isomorphisms of \mathcal{D} , is Borel.

Observe that if (\mathcal{C}_X, f, F) is a *good parametrization* of \mathcal{D} , then (X, f) is a good parametrization of \mathcal{C} in the sense of [19, Definition 2.1].

Definition 2.2.2. Let \mathcal{D} be a category and let (\mathcal{C}_X, f, F) and $(\mathcal{C}_{X'}, f', F')$ be good parametrizations of \mathcal{D} . A *morphism* from (\mathcal{C}_X, f, F) to $(\mathcal{C}_{X'}, f', F')$ is a triple (g, G, η) of maps $g: X \rightarrow X'$, $G: \mathcal{C}_X \rightarrow \mathcal{C}_{X'}$, and $\eta: X \rightarrow \mathcal{D}$, satisfying the following conditions:

- (1) The functions g and G are Borel;
- (2) $\eta(x)$ is an isomorphism from $f(x)$ to $(f' \circ g)(x)$ for every $x \in X$;
- (3) The pair $(f' \circ g, F' \circ G)$ is a semigroupoid morphism $\mathcal{C}_X \rightarrow \mathcal{D}$;
- (4) We have $s_{X'} \circ G = g \circ s_X$ and $r_{X'} \circ G = g \circ r_X$;
- (5) For every $x \in X$, the morphism $\eta(s(x))$ is an isomorphism from $F(x)$ to $(F' \circ G)(x)$;

(6) For every $a \in \mathcal{C}_X$

$$F'(G(a)) \circ \eta(s(a)) = \eta(r(a)) \circ F(a).$$

Two good parametrizations (\mathcal{C}_X, f, F) and $(\mathcal{C}_{X'}, f', F')$ of \mathcal{C} are said to be *equivalent* if there are isomorphisms from (\mathcal{C}_X, f, F) to $(\mathcal{C}_{X'}, f', F')$ and viceversa. It is not difficult to verify that if (\mathcal{C}_X, f, F) and $(\mathcal{C}_{X'}, f', F')$ are equivalent parametrizations of \mathcal{D} , then (X, f) and (X', f') are weakly equivalent parametrizations of \mathcal{D} in the sense of [19, Definition 2.1].

In the following, a good parametrization (\mathcal{C}_X, f, F) of \mathcal{D} will be denoted by \mathcal{C}_X for short.

2.3. The space $\mathcal{C}_{\widehat{\Xi}}$. We follow the notation in [19, Section 2.2], and denote by $\mathbb{Q}(i)$ the field of complex rationals. A $\mathbb{Q}(i)$ -*-algebra is an algebra over the field $\mathbb{Q}(i)$ endowed with an involution $x \mapsto x^*$. We define \mathcal{U} to be the $\mathbb{Q}(i)$ -*-algebra of noncommutative *-polynomials with coefficients in $\mathbb{Q}(i)$ and without constant term in the formal variables X_k for $k \in \omega$. If A is a C*-algebra, $\gamma = (\gamma_n)_{n \in \omega}$ is a sequence of elements of A , and $p \in \mathcal{U}$, we define $p(\gamma)$ to be the element of A obtained by evaluating p in A , where for every $k \in \omega$, the formal variables X_k and X_k^* are replaced by the elements γ_k and γ_k^* of A .

We denote by $\widehat{\Xi}$ the set of elements

$$A = (f, g, h, k, r) \in \omega^{\omega \times \omega} \times \omega^{\mathbb{Q}(i) \times \omega} \times \omega^{\omega \times \omega} \times \omega^{\omega} \times \mathbb{R}^{\omega}$$

that code on ω a structure of $\mathbb{Q}(i)$ -*-algebra A endowed with a norm satisfying the C*-identity. The completion \widehat{A} of ω with respect to such norm is a C*-algebra (denoted by $B(A)$ in [19, Subection 2.4]). It is not hard to check that $\widehat{\Xi}$ is a G_{δ} subspace of $\omega^{\omega \times \omega} \times \omega^{\mathbb{Q}(i) \times \omega} \times \omega^{\omega \times \omega} \times \omega^{\omega} \times \mathbb{R}^{\omega}$, and hence Polish with the subspace topology. As observed in [19, Subection 2.4], $\widehat{\Xi}$ can be thought of as a natural parametrization for *abstract C*-algebras*. We use the notation of [19, Subsection 2.4] to denote the operations on ω coded by an element $A = (f, g, h, k, r)$ of $\widehat{\Xi}$. We denote by d_A the metric on ω coded by A , which is given by

$$d_A(n, m) = \|n +_f (-1) \cdot_g m\|_r$$

for $n, m \in \omega$. We will also write $n +_A m$ for $n +_f m$, and similarly for g, h, k, r .

Definition 2.3.1. Suppose that $A = (f, g, h, k, r)$ and $A' = (f', g', h', k', r')$ are elements of $\widehat{\Xi}$, and that $\Phi = (\Phi_n)_{n \in \omega} \in (\omega^{\omega})^{\omega}$ is a sequence of functions from ω to ω . We say that Φ is a *code for a *-homomorphism* from \widehat{A} to \widehat{A}' if the following conditions hold:

- (1) The sequence $(\Phi_n(k))_{n \in \omega}$ is Cauchy uniformly in $k \in \omega$ with respect to the metric d_A , and in particular converges to an element $\widehat{\Phi}(k)$ of \widehat{A} ;

- (2) The map $k \mapsto \widehat{\Phi}(k)$ is a contractive $*$ -homomorphism of $\mathbb{Q}(i)$ - $*$ -algebras, and hence it induces a $*$ -homomorphism $\widehat{\Phi}$ from \widehat{A} to \widehat{A}' .

We say that Φ is a *code for an isomorphism* from \widehat{A} to \widehat{A}' if Φ is a code for a $*$ -homomorphism from \widehat{A} to \widehat{A}' , and $\widehat{\Phi}$ is an isomorphism. If Φ and Φ' are codes for $*$ -homomorphisms from \widehat{A} to \widehat{A}' and from \widehat{A}' to \widehat{A}'' respectively, we define their composition $\Phi' \circ \Phi$, which will be a code for a $*$ -homomorphism from \widehat{A} to \widehat{A}'' , by $(\Phi' \circ \Phi)_n = \Phi'_n \circ \Phi_n$ for $n \in \omega$.

Remark 2.3.2. It is easily checked that $\Phi' \circ \Phi \in (\omega^\omega)^\omega$ is a code for the $*$ -homomorphism $\widehat{\Phi'} \circ \widehat{\Phi}$ from \widehat{A} to \widehat{A}'' .

One can verify that the set $\mathcal{C}_{\widehat{\Xi}}$ of triples $(A, A', \Phi) \in \widehat{\Xi} \times \widehat{\Xi} \times (\omega^\omega)^\omega$ such that Φ is a code for a $*$ -homomorphism from \widehat{A} to \widehat{A}' , is Borel. We can regard $\mathcal{C}_{\widehat{\Xi}}$ as a standard semigroupoid having $\widehat{\Xi}$ as set of objects, where the composition of (A, A', Φ) and (A', A'', Φ') is $(A, A'', \Phi' \circ \Phi)$, and the source and range of (A, A', Φ) are A and A' respectively. The semigroupoid morphism $(A, A', \Phi) \mapsto (\widehat{A}, \widehat{A}', \widehat{\Phi})$ defines a parametrization of the category of C^* -algebras with $*$ -homomorphisms. It is easy to see that this is a good parametrization in the sense of Definition 2.2.2. In particular, the set $\text{Iso}_{\widehat{\Xi}}$ of elements (A, A', Φ) of $\widehat{\Xi} \times \widehat{\Xi} \times (\omega^\omega)^\omega$ such that Φ is a code for an isomorphism from \widehat{A} to \widehat{A}' , is Borel.

2.4. The space \mathcal{C}_{Ξ} . We denote by Ξ the G_δ subset of $\mathbb{R}^\mathcal{U}$ consisting of the nonzero functions $\delta: \mathcal{U} \rightarrow \mathbb{R}$ such that there exists a C^* -algebra A and a dense subset $\gamma = (\gamma_n)_{n \in \omega}$ of A , such that

$$\delta(p) = \|p(\gamma)\|.$$

It could be observed that, differently from [19, Subsection 2.3], we are not considering the function constantly equal to zero as an element of Ξ ; this choice is just for convenience and will play no role in the rest of the discussion. Observe that any element δ of Ξ determines a seminorm on the $\mathbb{Q}(i)$ - $*$ -algebra \mathcal{U} ; therefore one can consider the corresponding Hausdorff completion of \mathcal{U} . Denote by I_δ the ideal of \mathcal{U} given by

$$I_\delta = \{p \in \mathcal{U} : \delta(p) = 0\}.$$

Then \mathcal{U}/I_δ is a normed $\mathbb{Q}(i)$ - $*$ -algebra. Its completion is a C^* -algebra, which we shall denote by $\widehat{\delta}$. (Notice that what we denote by $\widehat{\delta}$, is denoted by $B(\delta)$ in [19, Subsection 2.3].)

Definition 2.4.1. Let δ and δ' be elements in Ξ , and let $\Phi = (\Phi_n)_{n \in \omega} \in (\mathcal{U}^\mathcal{U})^\omega$ be a sequence of functions from \mathcal{U} to \mathcal{U} . We say that Φ is a *code for a $*$ -homomorphism* from $\widehat{\delta}$ to $\widehat{\delta}'$, if

- (1) for every $p \in \mathcal{U}$, the sequence $(\Phi_n(p))_{n \in \omega}$ is Cauchy uniformly in $p \in \mathcal{U}$, with respect to the pseudometric $(q, q') \mapsto \delta(q - q')$ on \mathcal{U} , and in particular converges in $\widehat{\delta}$ to an element $\widehat{\Phi}(p)$;

- (2) $p \mapsto \widehat{\Phi}(p)$ is a morphism of $\mathbb{Q}(i)$ -*-algebras such that $\|\widehat{\Phi}(p)\| \leq \delta(p)$,
and hence induces a *-homomorphism from $\widehat{\delta}$ to $\widehat{\delta}'$.

Writing down explicit formulas defining a code for a *-homomorphism makes it clear that the set \mathcal{C}_{Ξ} of triples $(\delta, \delta', \Phi) \in \Xi \times \Xi \times (\mathcal{U}^{\mathcal{U}})^{\omega}$ such that Φ is a code for a *-homomorphism from $\widehat{\delta}$ to $\widehat{\delta}'$ is Borel. Suppose that Φ, Φ' are code for *-homomorphisms from δ to δ' and from δ' to δ'' . Similarly as in Subsection 2.3, it is easy to check that defining

$$(\Phi' \circ \Phi)_n = \Phi'_n \circ \Phi_n$$

for $n \in \omega$ gives a code for a *-homomorphism from δ to δ'' . This defines a standard Borel semigroupoid structure on \mathcal{C}_{Ξ} , such that the map $(\delta, \delta', \Phi) \mapsto (\widehat{\delta}, \widehat{\delta}', \widehat{\Phi})$ is a good standard Borel parametrization of the category of C^* -algebras.

2.5. The space $\mathcal{C}_{\Gamma(H)}$. Denote by $B_1(H)$ the unit ball of $B(H)$ with respect to the operator norm. Recall that $B_1(H)$ is a compact Hausdorff space when endowed with the weak operator topology. The standard Borel structure generated by the weak topology on $B_1(H)$ coincide with the Borel structure generated by several other operator topologies on $B_1(H)$, such as the σ -weak, strong, σ -strong, strong-*, and σ -strong-* operator topology; see [1, I.3.1.1]. Denote by $B_1(H)^{\omega}$ the product of countable many copies of $B_1(H)$, endowed with the product topology, and define $\Gamma(H)$ to be the Polish space obtained by removing from $B_1(H)^{\omega}$ the sequence constantly equal to 0. (The space $\Gamma(H)$ is defined similarly in [19, Subection 2.1]; the only difference is that here the sequence constantly equal to 0 is excluded for convenience.) Given an element γ in $\Gamma(H)$, denote by $C^*(\gamma)$ the C^* -subalgebra of $B(H)$ generated by $\{\gamma_n : n \in \omega\}$. As explained in [19, Subsection 2.1 and Remark 2.3], the space $\Gamma(H)$ can be thought of as a natural parametrization of *concrete C^* -algebras*.

Definition 2.5.1. Let γ and γ' be elements in $\Gamma(H)$, and let $\Phi = (\Phi_n)_{n \in \omega} \in (\mathcal{U}^{\mathcal{U}})^{\omega}$ be a sequence of functions from \mathcal{U} to \mathcal{U} . We say that Φ is a *code for a *-homomorphism* from $C^*(\gamma)$ to $C^*(\gamma')$, if

- (1) the sequence $(\Phi_n(p)(\gamma'))_{n \in \omega}$ of elements of $C^*(\gamma')$ is Cauchy uniformly in p , and hence converges to an element $\widehat{\Phi}(p(\gamma))$ of $C^*(\gamma')$;
- (2) the function $p(\gamma) \mapsto \widehat{\Phi}(p(\gamma))$ extends to a *-homomorphism from $C^*(\gamma)$ to $C^*(\gamma')$.

Again, it is easily checked that the set $\mathcal{C}_{\Gamma(H)}$ of triples (γ, γ', Φ) such that Φ is a code for a *-homomorphism from $C^*(\gamma)$ to $C^*(\gamma')$, is Borel. Moreover, one can define a standard Borel semigroupoid structure on $\mathcal{C}_{\Gamma(H)}$, in such a way that the map $(\gamma, \gamma', \Phi) \mapsto (C^*(\gamma), C^*(\gamma'), \widehat{\Phi})$ is a good parametrization of the category of C^* -algebras.

For future reference, we show in Lemma 2.5.2 below that in the parametrization $\mathcal{C}_{\Gamma(H)}$ one can compute a code for the inverse of an isomorphism in a Borel way.

Lemma 2.5.2. *There is a Borel map from $\text{Iso}_{\Gamma(H)}$ to $(\mathcal{U}^{\mathcal{U}})^{\omega}$, assigning to an element (γ, γ', Φ) of $\text{Iso}_{\Gamma(H)}$ a code $\text{Inv}(\gamma, \gamma', \Phi)$ for an isomorphism from $C^*(\gamma')$ to $C^*(\gamma)$ such that $\widehat{\text{Inv}(\gamma, \gamma', \Phi)} = \widehat{\Phi}^{-1}$.*

Proof. Observe that the set \mathcal{E} of tuples

$$((\gamma, \gamma', \Phi), p, n, q, N) \in \text{Iso}_{\Gamma(H)} \times \mathcal{U} \times \omega \times \mathcal{U} \times \omega$$

such that

$$\|q(\gamma') - \Phi_M(p)(\gamma')\| < \frac{1}{n},$$

and

$$\|\Phi_{M'}(p)(\gamma') - \Phi_M(p)(\gamma')\| < \frac{1}{n},$$

for every $M, M' \geq N$ is Borel. Therefore one can find Borel functions $(\xi, q, n) \mapsto p_{(\xi, p, n)}$ and $(\xi, p, n) \mapsto N_{(\xi, p, n)}$ from $\text{Iso}_{\Gamma(H)} \times \mathcal{U} \times \omega$ to \mathcal{U} and ω respectively such that

$$(\xi, q, n, p_{(\xi, q, n)}, N_{(\xi, q, n)}) \in \mathcal{E}$$

for every $(\xi, q, n) \in \text{Iso}_{\Gamma(H)} \times \mathcal{U} \times \omega$. Defining now $\text{Inv}(\xi)_n(q) = p_{(\xi, q, n)}$ for every $n \in \omega$ and $q \in \mathcal{U}$ one obtains a Borel map $\xi \mapsto \text{Inv}(\xi)$. Moreover,

$$\begin{aligned} & \left\| \text{Inv}(\xi)_n(q)(\gamma) - \widehat{\Phi}^{-1}(q(\gamma')) \right\| \\ & \leq \left\| p_{(\xi, q, n)}(\gamma) - \widehat{\Phi}^{-1}(\Phi_{N_{(\xi, q, n)}}(p)(\gamma')) \right\| + \frac{1}{n} \\ & = \left\| \widehat{\Phi}(p_{(\xi, q, n)}(\gamma)) - \Phi_{N_{(\xi, q, n)}}(p)(\gamma') \right\| + \frac{1}{n} \\ & \leq \frac{1}{2n}. \end{aligned}$$

This shows that $\text{Inv}(\xi)$ is a code for the inverse of $\widehat{\Phi}$. \square

2.6. Equivalence of \mathcal{C}_{Ξ} , \mathcal{C}_{Ξ} and \mathcal{C}_{Γ} . Recall that given an element δ of Ξ , we denote by I_{δ} the ideal of \mathcal{U} given by

$$I_{\delta} = \{p \in \mathcal{U} : \delta(p) = 0\}.$$

Theorem 2.6.1. *The good parametrizations \mathcal{C}_{Ξ} , \mathcal{C}_{Ξ} , and \mathcal{C}_{Γ} , of the category of C^* -algebras with $*$ -homomorphisms, are equivalent in the sense of Definition 2.2.2.*

Proof. We will show first that \mathcal{C}_{Ξ} and \mathcal{C}_{Ξ} are equivalent.

We start by constructing a morphism from \mathcal{C}_{Ξ} to \mathcal{C}_{Ξ} as in Definition 2.2.2 as follows. As in the proof of [19, Proposition 2.6], for every $n \in \omega$ define a Borel map $p_n : \Xi \rightarrow \mathcal{U}$, denoted $\delta \mapsto p_n^{\delta}$ for δ in Ξ , such that

$$\left\{ p_n^{\delta} + I_{\delta} : n \in \omega \right\}$$

is an enumeration of \mathcal{U}/I_δ for every $\delta \in \Xi$. For $\delta \in \Xi$, define a structure of C^* -normed $\mathbb{Q}(i)$ -*-algebra $A_\delta = (f_\delta, g_\delta, h_\delta, k_\delta, r_\delta)$ on ω by:

- $m +_{f_\delta} n = t$ whenever $p_m^\delta + p_n^\delta + I_\delta = p_t^\delta + I_\delta$;
- $q \cdot_{g_\delta} m = t$ whenever $q \cdot p_m^\delta + I_\delta = p_t^\delta + I_\delta$;
- $m \cdot_{h_\delta} n = t$ whenever $q_m^\delta q_n^\delta + I_\delta = q_t^\delta + I_\delta$;
- $m^{*k_\delta} = t$ whenever $(q_m^\delta)^* + I_\delta = q_t^\delta + I_\delta$;
- $\|m\|_{r_\delta} = \delta(q_m^\delta)$.

It is clear that the map $\delta \mapsto A_\delta$ is Borel. Moreover, for fixed δ in Ξ , the map $n \mapsto p_n^\delta + I_\delta$ is an isomorphism of normed $\mathbb{Q}(i)$ -*-algebras from A_δ onto \mathcal{U}/I_δ . We denote by $\eta_\delta: \widehat{A}_\delta \rightarrow \widehat{\delta}$ the induced isomorphism of C^* -algebras.

Now, if $\xi = (\delta, \delta', \Phi)$ belongs to \mathcal{C}_Ξ , define $\Psi_\xi \in (\mathcal{U}^\omega)^\omega$ by

$$(\Psi_\xi)_n(m) = k \text{ whenever } \Phi_n(p_m^\delta) + I_\delta = p_k^\delta + I_\delta,$$

for n, m and k in ω . It is not difficult to check that Ψ_ξ is a code for a *-homomorphism from A_δ to $A_{\delta'}$, and that the assignment $\xi \mapsto \Psi_\xi$ is Borel. Thus, the map from \mathcal{C}_Ξ to $\mathcal{C}_{\widehat{\Xi}}$ that assigns to the element $\xi = (\delta, \delta', \Phi)$ in \mathcal{C}_Ξ , the element $(A_\delta, A_{\delta'}, \Psi_\xi)$ of $\mathcal{C}_{\widehat{\Xi}}$, is Borel. Finally, it is easily verified that the map

$$\xi = (\delta, \delta', \Phi) \mapsto (\widehat{A}_\delta, \widehat{A}_{\delta'}, \widehat{\Psi}_\xi)$$

is a functor from \mathcal{C}_Ξ to the category of C^* -algebras. Moreover, if $\xi = (\delta, \delta', \Phi) \in \Xi$, then it follows from the construction that

$$\widehat{\Phi} \circ \eta_\delta = \eta_{\delta'} \circ \widehat{\Psi}_\xi.$$

We now proceed to construct morphism from $\mathcal{C}_{\widehat{\Xi}}$ to \mathcal{C}_Ξ . This will conclude the proof that $\mathcal{C}_{\widehat{\Xi}}$ and \mathcal{C}_Ξ are equivalent parametrizations according to Definition 2.2.2.

For $A \in \widehat{\Xi}$ and $p \in \mathcal{U}$, denote by p_A the evaluation of p in the $\mathbb{Q}(i)$ -*-algebra on ω coded by A , where the formal variable X_j is replaced by j for every $j \in \omega$. Write $A = (f, g, h, k, r)$, and define an element δ_A of Ξ by

$$\delta_A(p) = \|p_A\|_r,$$

for all p in \mathcal{U} . It is easily checked that the map $A \mapsto \delta_A$ is a Borel function from $\widehat{\Xi}$ to Ξ . For every $n \in \omega$, define a Borel map $p_n: \widehat{\Xi} \rightarrow \mathcal{U}$, denoted $A \mapsto p_n^A$ for A in $\widehat{\Xi}$, such that

$$\{p_n^A + I_{\delta_A} : n \in \omega\}$$

is an enumeration of \mathcal{U}/I_{δ_A} . The function $n \mapsto p_n^A + I_{\delta_A}$ induces an isomorphism of normed $\mathbb{Q}(i)$ -*-algebras, from ω with the structure coded by A , and \mathcal{U}/I_{δ_A} . One checks that this isomorphism induces a C^* -algebra isomorphism between \widehat{A} and $\widehat{\delta_A}$.

For $\xi = (A, A', \Psi) \in \mathcal{C}_{\widehat{\Xi}}$, define $\Psi_\xi \in (\mathcal{U}^\omega)^\omega$ by

$$(\Psi_\xi)_n(p) = q_m^A \text{ whenever } p + I_{\delta_A} = p_k^A + I_{\delta_A} \text{ and } \Psi_n(k) = m.$$

It can easily be checked that

- Ψ_ξ is a code for a $*$ -homomorphism from $\widehat{\delta}^A$ to $\widehat{\delta}_{A'}$,
- the map $\xi \mapsto \Psi_\xi$ is Borel, and
- $\widehat{\Psi}_\xi \circ \eta_A = \eta_{A'} \circ \widehat{\Psi}_\xi$.

This concludes the proof that \mathcal{C}_Ξ and $\mathcal{C}_{\widehat{\Xi}}$ are equivalent good parametrizations of the category of C^* -algebras.

We proceed to show that \mathcal{C}_Ξ and \mathcal{C}_Γ are equivalent parametrizations.

Denote by $\delta: \Gamma(H) \rightarrow \Xi$ and $\gamma: \Xi \rightarrow \Gamma(H)$ the Borel maps defined in the proof of [19, Proposition 2.7] witnessing the fact that Ξ and $\Gamma(H)$ are weakly equivalent parametrizations in the sense of [19, Definition 2.1]. It is straightforward to check that the maps $\Delta: \mathcal{C}_{\Gamma(H)} \rightarrow \mathcal{C}_\Xi$ and $\Gamma: \mathcal{C}_\Xi \rightarrow \mathcal{C}_{\Gamma(H)}$ given by

$$\Delta(\gamma, \gamma', \Phi) = (\delta_\gamma, \delta_{\gamma'}, \Phi) \quad \text{and} \quad \Gamma(\delta, \delta', \Psi) = (\gamma_\delta, \gamma_{\delta'}, \Psi)$$

are morphisms of good parametrizations, witnessing the facts that $\mathcal{C}_{\Gamma(H)}$ and \mathcal{C}_Ξ are equivalent. \square

2.7. Direct limits of C^* -algebras. An *inductive system* in the category of C^* -algebras is a sequence $(A_n, \varphi_n)_{n \in \omega}$, where for every n in ω , A_n is a C^* -algebra, and $\varphi_n: A_n \rightarrow A_{n+1}$ is a $*$ -homomorphism. The *inductive limit* of the inductive system $(A_n, \varphi_n)_{n \in \omega}$ is the C^* -algebra $\varinjlim (A_n, \varphi_n)$ defined as in [1, II.8.2]. It is verified in [19, Subsection 3.2] that the inductive limit of an inductive system of C^* -algebras can be computed in a Borel way. We report here, for the sake of completeness, a different proof.

We will work in the parametrization \mathcal{C}_Ξ of the category of C^* -algebras. In view of the equivalence of the parametrizations \mathcal{C}_Ξ , $\mathcal{C}_{\widehat{\Xi}}$, and $\mathcal{C}_{\Gamma(H)}$, the same result holds if one instead considers either one of the parametrizations $\mathcal{C}_{\widehat{\Xi}}$ or $\mathcal{C}_{\Gamma(H)}$.

Denote by $R_{dir}(\Xi)$ the set of sequences $(\delta_n, \Phi_n)_{n \in \omega} \in (\Xi \times (\mathcal{U}^\omega)^\omega)$ such that Φ_n is a code for a $*$ -homomorphism $\widehat{\delta}_n \rightarrow \widehat{\delta}_{n+1}$ for every $n \in \omega$. We can regard $R_{dir}(\Xi)$ as the standard Borel space parametrizing inductive systems of C^* -algebras. (Though the subscript in R_{dir} stands for “direct system”, we choose the term “inductive system” since we only deal with sequences. Since the notation R_{dir} was already introduced in [18], with the same meaning, we keep it despite calling its elements differently.)

Proposition 2.7.1. *There is a Borel map from $R_{dir}(\Xi)$ to Ξ that assigns to an element $(\delta_n, \Phi_n)_{n \in \omega}$ of $R_{dir}(\Xi)$ an element $\lambda_{(\delta_n, \Phi_n)_{n \in \omega}}$ of Ξ such that $\widehat{\lambda}_{(\delta_n, \Phi_n)_{n \in \omega}}$ is isomorphic to the inductive limit of the inductive system $(\widehat{\delta}_n, \widehat{\Phi}_n)_{n \in \omega}$. Moreover, for every $k \in \omega$ there is a Borel map from $R_{dir}(\Xi)$ to $(\mathcal{U}^\omega)^\omega$ that assigns to $(\delta_n, \Phi_n)_{n \in \omega}$ a code I_k for the canonical $*$ -homomorphism from $\widehat{\delta}_k$ to the inductive limit $\widehat{\lambda}_{(\delta_n, \Phi_n)_{n \in \omega}}$.*

Proof. Denote for $n \in \omega$ by \mathcal{U}_n the $\mathbb{Q}(i)$ -*-algebra of *-polynomials in the pairwise distinct noncommutative variables $(X_i^{(n)})_{i \in \omega}$. Similarly define \mathcal{U}_∞ to be the $\mathbb{Q}(i)$ -*-algebra of *-polynomials in the noncommutative variables $(X_i^{(n)})_{(i,n) \in \omega \times \omega}$. We will naturally identify \mathcal{U}_n as a $\mathbb{Q}(i)$ -*-subalgebra of \mathcal{U}_∞ , and define \mathcal{V}_n to be the $\mathbb{Q}(i)$ -*-subalgebra of \mathcal{U}_∞ generated by

$$\bigcup_{i \in n} \mathcal{U}_i$$

inside \mathcal{U}_∞ . Fix an element $(\delta_n, \Phi_n)_{k \in \omega}$ of $R_{dir}(\Xi)$. To simplify the notation we will assume that $\delta_n: \mathcal{U}_n \rightarrow \mathbb{R}$ for every $n \in \omega$, and $\Phi_n \in (\mathcal{U}_{n+1}^{\mathcal{U}_n})^\omega$. Correspondingly we will define a function $\lambda_{(\delta_n, \Phi_n)_{n \in \omega}}: \mathcal{U}_\infty \rightarrow \mathbb{R}$. Fix $n \in n' \in \omega$ and $k \in \omega$. Define

$$\Phi_{n',n,k}: \mathcal{V}_n \rightarrow \mathcal{U}_{n'}$$

to be the function obtained by freely extending the maps

$$(\Phi_{n'-1} \circ \cdots \circ \Phi_i)_k: \mathcal{U}_i \rightarrow \mathcal{U}_{n'}$$

for $i \in n$. Finally define for every $N \in \omega$ and $p \in \mathcal{V}_N \subset \mathcal{U}_\infty$

$$\lambda_{(\delta_n, \Phi_n)_{n \in \omega}}(p) = \lim_{n' > N} \lim_{k \rightarrow \infty} \delta_{n'}(\Phi_{n',N,k}(p)).$$

It is immediate to verify that the definition does not depend on N . Moreover $\lambda_{(\delta_n, \Phi_n)_{n \in \omega}}: \mathcal{U}_\infty \rightarrow \mathbb{R}$ define a seminorm on \mathcal{U}_∞ such that $\widehat{\lambda}_{(\delta_n, \Phi_n)_{n \in \omega}}$ is isomorphic to the direct limit of the inductive system $(\widehat{\delta}_n, \widehat{\Phi}_n)_{n \in \omega}$. If $N \in \omega$ and $\iota_N: \mathcal{U}_N \rightarrow \mathcal{U}_\infty$ denotes the inclusion map, and $I_N \in (\mathcal{U}_\infty^{\mathcal{U}_N})^\omega$ denotes the sequence constantly equal to ι_N , then I_N is a code for the canonical *-homomorphism from $\widehat{\delta}_k$ to the direct limit $\widehat{\lambda}_{(\delta_n, \Phi_n)_{n \in \omega}}$. \square

2.8. One sided intertwining.

Definition 2.8.1. Let $(A_n, \varphi_n)_{n \in \omega}$ and $(A'_n, \varphi'_n)_{n \in \omega}$ be inductive systems of C*-algebras. A sequence $(\psi_n)_{n \in \omega}$ of homomorphisms $\psi_n: A_n \rightarrow A'_n$ is said to be a *one sided intertwining* between $(A_n, \varphi_n)_{n \in \omega}$ and $(A'_n, \varphi'_n)_{n \in \omega}$, if the diagram

$$\begin{array}{ccccccc} A_0 & \xrightarrow{\varphi_0} & A_1 & \xrightarrow{\varphi_1} & A_2 & \xrightarrow{\varphi_2} & \cdots \\ \psi_0 \downarrow & & \psi_1 \downarrow & & \psi_2 \downarrow & & \\ A'_0 & \xrightarrow{\varphi'_0} & A'_1 & \xrightarrow{\varphi'_1} & A'_2 & \xrightarrow{\varphi'_2} & \cdots \end{array}$$

is commutative.

If $(\psi_n)_{n \in \omega}$ is a one sided intertwining between $(A_n, \varphi_n)_{n \in \omega}$ and $(A'_n, \varphi'_n)_{n \in \omega}$, then there is an inductive limit homomorphism

$$\psi = \varinjlim \psi_n: \varinjlim (A_n, \varphi_n) \rightarrow \varinjlim (A'_n, \varphi'_n)$$

that makes the diagram

$$\begin{array}{ccccccc}
A_0 & \xrightarrow{\varphi_0} & A_1 & \xrightarrow{\varphi_1} & A_2 & \xrightarrow{\varphi_2} & \cdots \longrightarrow \varinjlim (A_n, \varphi_n) \\
\psi_0 \downarrow & & \psi_1 \downarrow & & \psi_2 \downarrow & & \downarrow \psi \\
A'_0 & \xrightarrow{\varphi'_0} & A'_1 & \xrightarrow{\varphi'_1} & A'_2 & \xrightarrow{\varphi'_2} & \cdots \longrightarrow \varinjlim (A'_n, \varphi'_n)
\end{array}$$

commutative.

In this subsection, we verify that the inductive limit homomorphism $\varinjlim \psi_n$ can be computed in a Borel way. We will work in the parametrization \mathcal{C}_Ξ of C^* -algebras. In view of the equivalence between the parametrizations \mathcal{C}_Ξ , $\mathcal{C}_{\hat{\Xi}}$ and $\mathcal{C}_{\Gamma(H)}$, the same result holds if one instead uses $\mathcal{C}_{\hat{\Xi}}$ or $\mathcal{C}_{\Gamma(H)}$.

Define $R_{int}(\Xi)$ to be the Borel set of all elements

$$((\delta_n, \Phi_n)_{n \in \omega}, (\delta'_n, \Phi'_n)_{n \in \omega}, (\Psi_n)_{n \in \omega}) \in R_{dir}(\Xi) \times R_{dir}(\Xi) \times (\mathcal{U}^\omega)^\omega$$

such that Ψ_n is a code for a $*$ -homomorphism from $\hat{\delta}_n$ to $\hat{\delta}'_{n+1}$ satisfying

$$\hat{\Psi}_{n+1} \circ \hat{\Phi}_n = \hat{\Phi}'_n \circ \hat{\Psi}_n$$

for every $n \in \omega$. In other words, $(\Psi_n)_{n \in \omega}$ is a sequence of codes for a one sided intertwining between the inductive systems coded by $(\delta_n, \Phi_n)_{n \in \omega}$ and $(\delta'_n, \Phi'_n)_{n \in \omega}$.

Proposition 2.8.2. *There is a Borel map from $R_{int}(\Xi)$ to $(\mathcal{U}^\omega)^\omega$ assigning to an element*

$$((\delta_n, \Phi_n)_{n \in \omega}, (\delta'_n, \Phi'_n)_{n \in \omega}, (\Psi_n)_{n \in \omega})$$

of $R_{int}(\Xi)$, a code Λ for the corresponding inductive limit homomorphism between the inductive limits of the systems coded by $(\delta_n, \Phi_n)_{n \in \omega}$ and $(\delta'_n, \Phi'_n)_{n \in \omega}$.

Proof. We will use the same notation as in the proof of Proposition 2.7.1. Fix an element $((\delta_n, \Phi_n)_{n \in \omega}, (\delta'_n, \Phi'_n)_{n \in \omega}, (\Psi_n)_{n \in \omega})$ of $R_{int}(\Xi)$. As in the proof of Proposition 2.7.1 we will assume that for every $n \in \omega$

$$\begin{aligned}
\delta_n &: \mathcal{U}_n \rightarrow \mathbb{R}, \\
\delta'_n &: \mathcal{U}_n \rightarrow \mathbb{R}, \\
\Phi_n &\in (\mathcal{U}_{n+1}^{\mathcal{U}_n})^\omega
\end{aligned}$$

and

$$\Phi'_n \in (\mathcal{U}_{n+1}^{\mathcal{U}_n})^\omega.$$

Therefore

$$\Psi_n \in (\mathcal{U}_n^{\mathcal{U}_n})^\omega$$

for every $n \in \omega$. Similarly the codes $\lambda_{(\delta_n, \Phi_n)_{n \in \omega}}$ and $\lambda_{(\delta'_n, \Phi'_n)_{n \in \omega}}$ for the direct limits of the systems coded by $(\delta_n, \Phi_n)_{n \in \omega}$ and $(\delta'_n, \Phi'_n)_{n \in \omega}$ will be supposed to be functions from \mathcal{U}_∞ to \mathbb{R} . We will therefore define a code $\Lambda \in (\mathcal{U}_\infty^{\mathcal{U}_\infty})^\omega$ for the $*$ -homomorphism coded by $(\Psi_n)_{n \in \omega}$. Recall from the

proof of Proposition 2.7.1 the definition of \mathcal{V}_N and $\Phi_{n',N,k}: \mathcal{V}_N \rightarrow \mathcal{U}_{n'}$ for $N \in n' \in \omega$ and $k \in \omega$. Fix functions $\sigma_0, \sigma_1, \sigma_2: \omega \rightarrow \omega$ such that

$$n \mapsto (\sigma_0(n), \sigma_1(n), \sigma_2(n))$$

is a bijection from ω to $\omega \times \omega \times \omega$. Fix $N \in \omega$ and define for $p \in \mathcal{V}_{\sigma_0(N)}$

$$\Lambda_N(p) = (\Psi_{\sigma_1(N), \sigma_2(N)} \circ \Phi_{\sigma_1(N), \sigma_0(N), \sigma_2(N)})(p)$$

and

$$\Lambda_N(p) = 0$$

for $p \notin \mathcal{V}_{\sigma_0(N)}$. It is not difficult to check that the sequence $(\Lambda_N)_{N \in \omega} \in (\mathcal{U}_\infty)^\omega$ indeed defines a code for the inductive limit homomorphism defined by the sequence $(\hat{\Psi}_n)_{n \in \omega}$. \square

2.9. Direct limits of groups. We consider as *standard Borel space* of infinite countable groups the set \mathcal{G} of functions $f: \omega \times \omega \rightarrow \omega$ such that the identity $n \cdot_f m = f(n, m)$ for $n, m \in \omega$, defines a group structure on ω . We consider \mathcal{G} as a Borel space with respect to the Borel structure inherited from $\omega^{\omega \times \omega}$; such Borel structure is standard, since \mathcal{G} is a Borel subset of $\omega^{\omega \times \omega}$. In the following, we will identify a group G and its code as an element of $\omega^{\omega \times \omega}$.

It is not difficult to check that most commonly studied classes of groups correspond to Borel subsets of \mathcal{G} . In particular we will denote by \mathcal{AG} Borel set of abelian groups, and by \mathcal{AG}_{TF} Borel set of torsion free abelian groups.

Let G be a discrete group and let α be an endomorphism of G . We will denote by $G_\infty = \varinjlim (G, \alpha)$ the inductive limit of the inductive system

$$G \xrightarrow{\alpha} G \xrightarrow{\alpha} \cdots \longrightarrow G_\infty.$$

For n in ω , denote by $\varphi_n: G \rightarrow G_\infty$ the canonical group homomorphism obtained by regarding G as the n -th stage of the inductive system above. Denote by α_∞ the unique automorphism of G_∞ such that $\alpha_\infty \circ \varphi_{n+1} = \varphi_n$ for every $n \in \omega$.

Denote by $End_{\mathcal{G}}$ the set of all pairs $(G, \alpha) \in \mathcal{G} \times \omega^\omega$, such that α is an injective endomorphism of G with respect to the group structure on ω coded by G , and note that $End_{\mathcal{G}}$ is Borel. Similarly define $DLim_{\mathcal{G}}$ to be the set of pairs $(G, \alpha) \in End_{\mathcal{G}}$ such that the direct limit $\varinjlim (G, \alpha)$ is infinite.

Proposition 2.9.1. *The set $DLim_{\mathcal{G}}$ is a Borel subset of $End_{\mathcal{G}}$. Moreover there is a Borel map from $DLim_{\mathcal{G}}$ to $End_{\mathcal{G}}$ that assigns to $(G, \alpha) \in DLim_{\mathcal{G}}$ the pair $(\varinjlim (G, \alpha), \alpha_\infty)$.*

Proof. Let (G, α) be an element in $End_{\mathcal{G}}$. Consider the equivalence relation \sim_α on $\omega \times \omega$ defined by

$$(x, i) \sim_\alpha (y, j) \text{ iff there exists } k \geq \max\{i, j\} \text{ with } \alpha^{k-i}(x) = \alpha^{k-j}(y).$$

Observe that $(G, \alpha) \in DLim_{\mathcal{G}}$ iff \sim_α has infinitely many classes. Therefore $DLim_{\mathcal{G}}$ is a Borel subset of \mathcal{G} by [31, Theorem 18.10]. Suppose now that

$(G, \alpha) \in D\text{Lim}_{\mathcal{G}}$. Consider the lexicographic order $<_{lex}$ on $\omega \times \omega$, and define the injective function $\eta_\alpha: \omega \rightarrow \omega \times \omega$ recursively on n as follows. Set $\eta_\alpha(0) = (0, 0)$, and for $n > 0$, define $\eta_\alpha(n)$ to be the $<_{lex}$ -minimum element (m, i) of $\omega \times \omega$ such that for every $k \in n$, we have $\eta_\alpha(k) \not\prec_\alpha (m, i)$. (Observe that the set of such elements is nonempty since we are assuming that \sim_α has infinitely many classes.) Define the group operation on ω by

$$n_0 \cdot_{G_\infty} n_1 = n$$

whenever there are $m_0, m_1, m, i_0, i_1, i, \tilde{i} \in \omega$ satisfying:

- $\eta_\alpha(n_0) = (m_0, i_0)$;
- $\eta_\alpha(n_1) = (m_1, i_1)$;
- $\eta(n) = (m, i)$;
- $\max\{i_0, i_1\} = \tilde{i}$;
- $(\alpha^{\tilde{i}-i_0}(m_0) \cdot_G \alpha^{\tilde{i}-i_1}(m_1), \tilde{i}) \sim (m, i)$.

Define the function $\alpha_\infty: \omega \rightarrow \omega$ by $\alpha_\infty(n) = n'$ if and only if there are $m, i, m', i' \in \omega$ such that:

- $\eta_\alpha(n) = (m, i)$;
- $\eta_\alpha(n') = (m', i')$;
- $(\alpha(m), i) \sim (m', i')$.

It is not difficult to check that G_∞ is the direct limit $\varinjlim (G, \alpha)$, and α_∞ is the automorphism of $\varinjlim (G, \alpha)$ corresponding to the endomorphism α of G . Moreover the function $(G, \alpha) \mapsto (G_\infty, \alpha_\infty)$ is Borel by construction. \square

2.10. Borel version of the Nielsen-Schreier theorem. The celebrated Nielsen-Schreier theorem [42, 54] asserts that a subgroup of a countable discrete free group is free. In this subsection we will prove a *Borel version* of such theorem, to be used in the proof of Lemma 2.11.1. This will be obtained by analyzing Schreier's proof of the theorem, as presented in [29, Chapter 2].

Denote by F the (countable) set of *reduced* words in the indeterminates x_n for $n \in \omega$ ordered lexicographically. We can identify the free group on countable many generators with F with the operation of *reduced concatenation* of words. It is immediate to check that the set $\mathcal{S}(F)$ of $H \in 2^\omega$ such that H is a subgroup of F is Borel.

Lemma 2.10.1. *There is a Borel function $H \mapsto B_H$ from $\mathcal{S}(F)$ to 2^F such that L_H is a set of free generators for H for every $H \in \mathcal{S}(F)$.*

Proof. Suppose that $H \in \mathcal{S}(F)$. If $a \in F$ denote by $\phi_H(a)$ the $<$ -minimal element of the coset Ha , where $<$ is the lexicographic order of F . Observe that $\phi_H(a) \leq b$ iff there is $b' \leq b$ such that $b'a^{-1} \in H$. This shows that the map

$$\begin{aligned} \mathcal{S}(F) &\rightarrow F^F \\ H &\mapsto \phi_H \end{aligned}$$

is Borel. Define B_H to be the set containing

$$\phi_H(a) x_n \phi_H(\phi_H(a) x_n)^{-1}$$

for $n \in \omega$ and $a \in F$ such that $\phi_H(a) x_n \neq \phi_H(c)$ for every $c \in F$. It is clear that the map $H \mapsto B_H$ is Borel. Moreover it can be shown as in [29, Chapter 2, Lemmas 3,4,5] that B_H is a free set of generators of H . \square

Suppose now that \mathbb{F}_ω is an element of \mathcal{G} representing the group of countably many generators, and $\mathcal{S}(F)$ is the Borel set of $H \in 2^\omega$ such that H is a subgroup of \mathbb{F}_ω . Proposition can be seen as just a reformulation of Lemma

Proposition 2.10.2. *There is a Borel map $H \mapsto B_H$ from $\mathcal{S}(F)$ to 2^ω that assigns to $H \in \mathcal{S}(F)$ a free set of generators of H .*

2.11. An exact sequence. The following lemma asserts that the construction of [49, Proposition 3.5] can be made in a Borel way.

Lemma 2.11.1. *There is a Borel function from \mathcal{AG} to $\mathcal{AG}_{TF} \times \omega^\omega$ that assigns to an abelian group G , a pair (H, α) , where H is a torsion free abelian group, and α is an automorphism of H such that*

$$H / (\text{id}_H - \alpha)[H] \cong G.$$

Proof. Denote by $\mathbb{F}_{\omega \times \omega}$ the free group with generators $x_{n,m}$ for $(n,m) \in \omega \times \omega$, suitably coded as an element of the standard Borel spaces of discrete groups \mathcal{AG} . Given an element $G \in \mathcal{AG}$, denote by N_G the subset of ω coding the kernel of the homomorphism from $\mathbb{F}_{\omega \times \omega}$ to G obtained by sending $x_{n,m}$ to n if $m = 0$, and to zero otherwise. In view of Proposition 2.10.2 one can find a Borel map

$$\begin{aligned} \mathcal{AG} &\rightarrow \omega^\omega \\ G &\mapsto x^G \end{aligned}$$

such that $x^G = (x_n^G)_{n \in \omega}$ is an enumeration of a free set of generators of N_G . Define an injective endomorphism δ_G of $\mathbb{F}_{\omega \times \omega}$ by

$$\delta_G(x_{n,m}) = \begin{cases} x_{n,m+1} & \text{if } m \neq -1, \\ x_n^G & \text{otherwise.} \end{cases}$$

Let $\beta_G: \mathbb{F}_{\omega \times \omega} \rightarrow \mathbb{F}_{\omega \times \omega}$ be $\beta_G = \text{id}_{\mathbb{F}_{\omega \times \omega}} - \delta_G$. By construction, the map $G \mapsto \beta_G$ is Borel. From now on we fix a group G , and abbreviate β_G to just β .

By Proposition 2.9.1, the inductive limit group $G_\infty = \varinjlim (G, \beta)$ and the automorphism $\beta_\infty = \varinjlim \beta$ can be constructed in a Borel way from G and β . We take $H = G_\infty$ and $\alpha = \beta_\infty$. It can now be verified, as in the proof of [49, Proposition 3.5], that G is isomorphic to the quotient of H by the image of $\text{id}_H - \alpha$. Moreover, it follows that the map $G \mapsto (H)$ is Borel. This finishes the proof. \square

3. COMPUTING REDUCED CROSSED PRODUCTS

The goal of this section is to show that the reduced crossed product of a C^* -algebra by an action of a discrete group can be computed in a Borel way. We begin by recalling the construction of the reduced crossed product.

3.1. The reduced crossed product. Suppose that A is a C^* -algebra, and α is an action of a countable discrete group G on A . We recall here the construction of the reduced crossed product $A \rtimes_{\alpha,r} G$ of A by α . Denote by $A[G]$ the *skew group algebra*. This is the complex $*$ -algebra

$$A[G] = \left\{ \sum_{g \in G} a_g g : a_g \in A, a_g = 0 \text{ for all but finitely many } g \in G \right\}.$$

The product on $A[G]$ is defined by twisted convolution

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{h \in G} b_h h \right) = \sum_{g, h \in G} a_g \alpha_g(b_h) gh.$$

The involution in $A[G]$ is given by

$$\left(\sum_{g \in G} b_g g \right)^* = \sum_{g \in G} \alpha_g(b_{g^{-1}}^*) g.$$

Suppose that H is a Hilbert space. A *covariant representation* of α on H is a pair (π, v) where

- (1) π is a representation of A on H , and
- (2) v is a unitary representation of G on H such that

$$v(g)\pi(a)v(g)^* = \pi(\alpha_g(a))$$

for every $g \in G$ and $a \in A$.

The *integrated form* of the covariant representation (π, v) is the $*$ -homomorphism $\pi \rtimes v$ from the twisted group algebra $A[G]$ to $B(H)$ defined by

$$(\pi \rtimes v) \left(\sum_{g \in G} a_g g \right) = \sum_{g \in G} \pi(a_g) v_g.$$

Let now ρ be a representation of A on H_0 . The *regular covariant representation* of α associated with ρ , is the covariant representation $(\pi_{\rho, \alpha}, v_{\rho, \alpha})$ of α on

$$H = \ell^2(G, H_0).$$

defined as follows: For $a \in A, g, h \in G$ and $\xi \in H$, set

$$(\pi_{\rho, \alpha}(a)\xi)(g) = \rho(\alpha_{g^{-1}}(a))\xi(g).$$

and

$$(v_{\rho, \alpha})_g(\xi)(h) = \xi(g^{-1}h).$$

Observe that, if $\lambda: G \rightarrow \ell^2(G)$ denotes the left regular representation of G , then the unitary operator $(v_{\rho,a})_g$ on $\ell^2(G, H)$ can be identified with

$$\lambda_g \otimes \text{id}_{H_0}$$

under the natural identification of $\ell^2(G, H_0)$ with $\ell^2(G) \otimes H_0$. The integrated form of $(\pi_{\rho,\alpha}, v_{\rho,\alpha})$ will be called the *regular representation* of $A[G]$ associated with ρ .

Definition 3.1.1. Suppose that α is an action of a discrete group G on A . For $a \in A[G]$, set

$$\|a\|_r = \sup \{ \|(\pi \rtimes v)(a)\| : (\pi, v) \text{ is a regular covariant representation} \},$$

and define the *reduced crossed product* of A by α , denoted $A \rtimes_{\alpha,r} G$, to be the completion of $A[G]$ with respect to the norm $\|\cdot\|_r$.

It is shown in [44, Theorem 7.7.5] that if ρ is a faithful representation of A on H_0 , then for any action α of a discrete group on A , the integrated form of the regular covariant representation of $(\pi_{\rho,\alpha}, v_{\rho,\alpha})$ on $\ell^2(G, H_0)$ associated with ρ induces a faithful representation of the reduced crossed product $A \rtimes_{\alpha,r} G$ on $\ell^2(G, H_0)$. Equivalently,

$$\|x\|_r = \|(\pi_{\rho,\alpha} \rtimes v_{\rho,\alpha})(x)\|$$

for every $x \in A[G]$.

One can also consider the completion of $A[G]$ with respect to the C^* -norm obtained as in 3.1.1 but considering the supremum over *all* covariant representations of α . One thus obtains the so called *full crossed product* $A \rtimes_{\alpha} G$.

Both full and reduced crossed products are C^* -algebras encoding information about the action α . For many purposes reduced crossed products are far better behaved and easier to understand than full ones. It is a standard fact in the theory of crossed products that if G is amenable, then full and reduced crossed products agree. See [60] for more details. In the following we will consider exclusively reduced crossed products

Similar notions can be defined for actions of locally compact group on C^* -algebras. More details can be found in [1, Section II.10].

3.2. Parametrizing actions of discrete groups on C^* -algebras. We proceed to construct a standard Borel parametrization of the space of all actions of discrete groups on C^* -algebras. For convenience, we will work using the parametrization $\Gamma(H)$ of C^* -algebras. In view of the weak equivalence of Ξ , $\widehat{\Xi}$, and $\Gamma(H)$, similar statements will hold for the parametrizations Ξ and $\widehat{\Xi}$.

Definition 3.2.1. Let γ be an element of $\Gamma(H)$, and G be an element of \mathcal{G} . Suppose that $\Phi = (\Phi_{m,n})_{(m,n) \in \omega \times \omega} \in (\mathcal{U}^{\mathcal{U}})^{\omega \times \omega}$ is an $(\omega \times \omega)$ -sequence

of functions from \mathcal{U} to \mathcal{U} . We say that Φ is a *code for an action of G on $C^*(\gamma)$* , if the following conditions hold:

- (1) for every $m \in \omega$, the sequence $(\Phi_{m,n})_{n \in \omega} \in (\mathcal{U}^{\mathcal{U}})^{\omega}$ is a code for an automorphism $\widehat{\Phi}_m$ of $C^*(\gamma)$,
- (2) $\Phi_{0,n}(m) = m$ for every $n, m \in \omega$, and
- (3) the function $m \mapsto \widehat{\Phi}_m$ is an action of G on $C^*(\gamma)$, this is,

$$\widehat{\Phi}_m \circ \widehat{\Phi}_k = \widehat{\Phi}_n$$

whenever $(m, k, n) \in G$.

It is easy to verify that any action of G on $C^*(\gamma)$ can be coded in a similar fashion. Moreover, the set $\text{Act}_{\Gamma(H)}$ of triples $(G, \gamma, \Phi) \in \mathcal{G} \times \Gamma(H) \times (\mathcal{U}^{\omega})^{\omega \times \omega}$ such that Φ is a code for an action of G on $C^*(\gamma)$, is a Borel subset of $\mathcal{G} \times \Gamma(H) \times (\mathcal{U}^{\omega})^{\omega \times \omega}$. We will regard $\text{Act}_{\Gamma(H)}$ as the parametrizing standard Borel space of actions of discrete groups on C^* -algebras.

3.3. Computing the reduced crossed product. We are now ready to prove that the reduced crossed product of a C^* -algebra by an action of a countable discrete group can be computed in a Borel way.

Proposition 3.3.1. *Let H be a separable Hilbert space. Then there is a Borel map $(G, \gamma, \Phi) \mapsto \delta_{(G, \gamma, \Phi)}$ from $\text{Act}_{\Gamma(H)}$ to $\Gamma(H)$ such that $C^*(\delta_{(G, \gamma, \Phi)}) \cong C^*(\gamma) \rtimes_{\widehat{\Phi}}^r G$. In other words, there is a Borel way to compute the code of the reduced crossed product of separable C^* -algebras by countable discrete groups.*

Proof. Denote by $\{e_k : k \in \omega\}$ the canonical basis of ℓ_2 . Let (G, γ, Φ) be an element of $\text{Act}_{\Gamma(H)}$. Define the element $\delta_{(G, \gamma, \Phi)}$ of $\Gamma(H)$ as follows. Given m in ω , denote by $m' \in \omega$ the inverse of m in G . Now set

$$(\delta_{(G, \gamma, \Phi)})_n(\xi \otimes m) = \begin{cases} \lim_{k \rightarrow +\infty} \gamma_{\Phi_{(m', k)}(r)} & \text{if } n = 2r, \text{ where } (n, m, k) \in G, \\ \xi \otimes k & \text{otherwise,} \end{cases}$$

for all ξ in H and all m in ω . The fact that $C^*(\delta_{(G, \gamma, \Phi)}) \cong C^*(\gamma) \rtimes_{\widehat{\Phi}, r}^r G$ follows from [44, Theorem 7.7.5]. Moreover, the map $(G, \gamma, \Phi) \mapsto \delta_{(G, \gamma, \Phi)}$ is Borel by construction and by [19, Lemma 3.4]. \square

Proposition 3.3.1 above answers half of [19, Problem 9.5(ii)]. It is not clear how to treat the case of full crossed products, even in the special case when the algebra is \mathbb{C} .

3.4. Crossed products by a single automorphism. Any automorphism α of a C^* -algebra A naturally induces an action of the group of integers \mathbb{Z} on A by $n \mapsto \alpha^n = \alpha \circ \dots \circ \alpha$.

In this subsection, we want to show that the crossed product of a C^* -algebra by a single automorphism, when regarded as an action of \mathbb{Z} , can be

computed in a Borel way. In view of the equivalence of the good parametrizations \mathcal{C}_Ξ , $\mathcal{C}_{\hat{\Xi}}$, and $\mathcal{C}_{\Gamma(H)}$, we can work in any of these. For convenience, we consider the parametrization $\mathcal{C}_{\Gamma(H)}$.

Let us denote by $\text{Aut}_{\Gamma(H)}$ the set of pairs (γ, Φ) in $\Gamma(H) \times (\mathcal{U}^\omega)^\omega$ such that Φ is a code for an automorphism of $C^*(\gamma)$. It is immediate to check that such set is Borel. We can regard $\text{Aut}_{\Gamma(H)}$ as the standard Borel space of automorphisms of C^* -algebras.

Lemma 3.4.1. *There is a Borel map from $\text{Aut}_{\Gamma(H)}$ to $\text{Act}_{\Gamma(H)}$ that assigns to an element (γ, Φ) in $\text{Aut}_{\Gamma(H)}$, a code for the action of \mathbb{Z} on $C^*(\gamma)$ associated with the automorphism coded by Φ .*

Proof. In the parametrization \mathcal{G} of discrete groups described before, the group of integers \mathbb{Z} is coded, for example, by the element $f_{\mathbb{Z}}$ of $\omega^{\omega \times \omega}$ given by

$$\begin{aligned} f_{\mathbb{Z}}(2n, 2m) &= 2(n + m) \\ f_{\mathbb{Z}}(2n - 1, 2m - 1) &= 2(n + m) - 1 \\ f_{\mathbb{Z}}(2n - 1, m) &= f_{\mathbb{Z}}(m, 2n - 1) = 2(n - m) - 1 \\ f_{\mathbb{Z}}(2m - 1, n) &= f_{\mathbb{Z}}(n, 2m - 1) = 2(n - m) \\ f_{\mathbb{Z}}(k, 0) &= f_{\mathbb{Z}}(0, k) = k \end{aligned}$$

for $n, m, k \in \omega$ with $n, m \geq 1$. Recall that by Lemma 2.5.2 there is a Borel map $\xi \mapsto \text{Inv}(\xi)$ from $\text{Iso}_{\Gamma(H)}$ to $(\mathcal{U}^\omega)^\omega$ such that if $\xi = (\gamma, \gamma', \Phi)$, then $\text{Inv}(\xi)$ is a code for the inverse of the $*$ -isomorphism coded by Φ . Suppose now that $(\gamma, \Phi) \in \text{Aut}_{\Gamma(H)}$. We want to define a code Ψ for the action of \mathbb{Z} on $C^*(\gamma)$ induced by $\hat{\Phi}$. For $n, m \in \omega$ with $m \geq 1$ define

$$\Psi_{0,n}(k) = k,$$

$$\Psi_{2m,n} = \overbrace{\Phi_n \circ \Phi_n \cdots \circ \Phi_n}^{m \text{ times}},$$

and

$$\Psi_{2m+1,n} = \text{Inv}(\gamma, \gamma, \Psi_m).$$

Observe that $(f_{\mathbb{Z}}, A, \Psi)$ is a code for the action of \mathbb{Z} associated with the automorphism $\hat{\Phi}$ of $C^*(\gamma)$. It is not difficult to verify that the map assigning $(f_{\mathbb{Z}}, A, \Psi)$ to (A, Φ) is Borel. We omit the details. \square

Corollary 3.4.2. *Given a C^* -algebra A and an automorphism α of A , there is a Borel way to compute the crossed product $A \rtimes_\alpha \mathbb{Z}$.*

Proof. Note that the group of integers \mathbb{Z} is amenable, so full and reduced crossed products coincide. The result now follows immediately from Lemma 3.4.1 together with Proposition 3.3.1. \square

3.5. Crossed product by an endomorphism. We now turn to crossed products by injective, corner endomorphisms, as introduced by Paschke in [43], building on previous work of Cuntz in [10]. Although there are more general theories for such crossed products allowing arbitrary endomorphisms of C^* -algebras (see, for example, [17]), the endomorphisms considered by Paschke will suffice for our purposes. We begin by presenting the precise definition of a corner endomorphism. Throughout this subsection, we fix a unital C^* -algebra A .

Definition 3.5.1. An endomorphism $\rho: A \rightarrow A$ is said to be a *corner endomorphism* if $\rho(A)$ is a corner of A , this is, if there exists a projection p in A such that $\rho(A) = pAp$.

Since A is unital, if $\rho: A \rightarrow A$ is a corner endomorphism and $\rho(A) = pAp$ for some projection p in A , then we must have $p = \rho(1)$. Let us observe for future reference that the set $CorEnd_\Gamma$ of pairs $(\gamma, \Phi) \in \Gamma \times (\mathcal{U}^\mathcal{U})^\omega$ such that $C^*(\gamma)$ is unital and Φ is a code for an injective corner endomorphism of $C^*(\gamma)$ is Borel. By [19, Lemma 3.14] the set Γ_u of $\gamma \in \Gamma$ such that $C^*(\gamma)$ is unital is Borel. Moreover, there is a Borel map $un: \Gamma_u \rightarrow B_1(H)$ such that $un(\gamma)$ is the unit of $C^*(\gamma)$ for every $\gamma \in \Gamma_u$. If now $\gamma \in \Gamma_u$ and $\Phi \in (\mathcal{U}^\mathcal{U})^\omega$, then Φ is a code for an injective corner endomorphism of $C^*(\gamma)$ if and only if Φ is a code for an endomorphism of A (which is a Borel condition, as observed in Subsection 2.5), and for every $p \in \mathcal{U}$ and $n \in \omega$ there is $m_0 \in \omega$ and $q \in \mathcal{U}$ such that for every $m \geq m_0$

$$\|\Phi_m(p)(\gamma)\| \geq \|p(\gamma)\| - \frac{1}{n}$$

and

$$\|un(\gamma)p(\gamma)un(\gamma) - \Phi_m(q)(\gamma)\| \leq \frac{1}{n}.$$

Let ρ be an injective corner endomorphism of A . The crossed product $A \rtimes_\rho \mathbb{N}$ of A by ρ is implicitly defined in [43] as the universal C^* -algebra generated by a unital copy of A together with an isometry S , subject to the relation

$$SaS^* = \rho(a)$$

for all a in A . Suppose that s is an isometry of A . Notice that the endomorphisms $a \mapsto sas^*$ is injective and its range is the corner $(ss^*)A(ss^*)$ of A .

Instead of using this construction, which involves universal C^* -algebras on generators and relations, we will use the construction of the endomorphism crossed product described by Stacey in [57]. Stacey's picture has the advantage that, given what we have proved so far, it will be relatively easy to conclude that crossed products by injective corner endomorphisms can be computed in a Borel way.

We proceed to describe Stacey's construction.

Definition 3.5.2. Let $\rho: A \rightarrow A$ be an injective corner endomorphism. Consider the inductive system $(A, \alpha)_{n \in \omega}$ (the same algebra and same connecting maps throughout the sequence), and denote by A_∞ its inductive limit, and by $\iota_{n,\infty}: A \rightarrow A_\infty$ the canonical map into the inductive limit. There is a commutative diagram

$$\begin{array}{ccccccc} A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots & \longrightarrow & A_\infty \\ \alpha \downarrow & & \alpha \downarrow & & & & \downarrow \alpha_\infty \\ A & \xrightarrow{\alpha} & A & \xrightarrow{\alpha} & \cdots & \longrightarrow & A_\infty. \end{array}$$

It is not hard to check that the corresponding endomorphism $\alpha_\infty: A_\infty \rightarrow A_\infty$ of the inductive limit is an automorphism. Denote by e the projection of A_∞ corresponding to the unit of A . The (endomorphism) crossed product of A by ρ is the corner $e(A_\infty \rtimes_{\alpha_\infty} \mathbb{Z})e$ of the (automorphism) crossed product $A_\infty \rtimes_{\alpha_\infty} \mathbb{Z}$.

As mentioned before, this construction of the crossed product of a C^* -algebra by an endomorphism makes it apparent that it can be computed in a Borel way. In fact, we have verified in Proposition 2.8.2, that the limit of a one sided intertwining can be computed in a Borel way, and in Corollary 3.4.2, that the crossed product of a C^* -algebra by an automorphism can be computed in a Borel way. Moreover, it is shown in [19, Lemma 3.14], that one can select in a Borel way the unit of a unital C^* -algebra. The only missing ingredient in the construction is taking a corner by a projection, which is shown to be Borel in the following lemma. We will work, for convenience, in the parametrization $\Gamma(H)$ of C^* -algebras.

Lemma 3.5.3. *The set $\Gamma_{proj}(H)$ of pairs (γ, e) in $\Gamma(H) \times B(H)$ such that e is a nonzero projection in $C^*(\gamma)$, is Borel. Moreover, there is a Borel map $(\gamma, e) \mapsto c_{\gamma,e}$ from $\Gamma_{proj}(H)$ to $\Gamma(H)$ such that $C^*(c_{\gamma,e})$ is the corner $eC^*(\gamma)e$ of $C^*(\gamma)$.*

Proof. Enumerate a dense subset $\{\xi_n: n \in \omega\}$ of H , and let (γ, e) be an element in $\Gamma(H) \times B(H)$. Observe that (γ, e) belongs to $\Gamma_{proj}(H)$ if and only if the following conditions hold:

- (1) The element e is a projection, this is, for every $n, k, t \in \omega$,

$$\langle (e - e^*) \xi_k, \xi_t \rangle < \frac{1}{n+1} \quad \text{and} \quad \langle (e^2 - e) \xi_k, \xi_t \rangle < \frac{1}{n+1};$$

- (2) The element e is non-zero, this is, there are $k, n, t \in \omega$ such that

$$\langle e \xi_k, \xi_t \rangle > \frac{1}{n+1};$$

- (3) The element e is in $C^*(\gamma)$, this is, for every $n \in \omega$ there is $p \in \mathcal{U}$ such that

$$\langle (p(\gamma) - e) \xi_m, \xi_t \rangle < \frac{1}{n+1},$$

for every $m, t \in \omega$.

This shows that $\Gamma_{proj}(H)$ is a Borel subset of $\Gamma(H) \times B(H)$. Observe now that by setting

$$(c_{\gamma,e})_n = e\gamma_n e$$

for every $n \in \omega$, one obtains an element $c_{\gamma,e}$ of $\Gamma(H)$ such that $C^*(c_{\gamma,e}) = eC^*(\gamma)e$. It is immediate to check that the map $(\gamma, e) \mapsto c_{\gamma,e}$ is Borel. \square

We have thus proved the following.

Corollary 3.5.4. *Given a unital C^* -algebra A and an injective corner endomorphism ρ of A , there is a Borel way to compute a code for the crossed product of A by ρ .*

More precisely, there is a Borel map

$$\begin{aligned} CorEnd_{\Gamma} &\rightarrow \Gamma \\ (\gamma, \Phi) &\mapsto \delta_{\gamma, \Phi} \end{aligned}$$

such that $C^(\delta_{\gamma, \Phi}) \cong C^*(\gamma) \rtimes_{\Phi} \mathbb{N}$, where, as before, $CorEnd_{\Gamma}$ is the Borel space of pairs (γ, Φ) in $\Gamma \times (\mathcal{U}^{\omega})^{\omega}$ such that $C^*(\gamma)$ is unital and Φ is a code for an injective corner endomorphism of $C^*(\gamma)$.*

Proof. Combine Lemma 3.5.3 with Proposition and Corollary 3.4.2. \square

4. BOREL SELECTION OF AF-ALGEBRAS

4.1. Bratteli diagrams. We refer the reader to [16] for the standard definition of a Bratteli diagram. We will identify Bratteli diagrams with elements

$$(l, (w_n)_{n \in \omega}, (m_n)_{n \in \omega}) \in \omega^{\omega} \times (\omega^{\omega})^{\omega} \times (\omega^{\omega \times \omega})^{\omega}$$

such that for every $i, j, n, m \in \omega$, the following conditions hold:

- (1) $l(0) = 1$;
- (2) $w_0(0) = 1$;
- (3) $w_n(i) > 0$ if and only if $i \in l(n)$;
- (4) $m_n(i, j) = 0$ whenever $i \geq l(n)$ or $j \geq l(n+1)$
- (5) With $k_n = i$,

$$w_n(i) = \sum_{k_j \in l(j), 1 \leq j < n} \prod_{t=0}^{n-1} m_t(k_t, k_{t+1}).$$

We denote by \mathcal{BD} the Borel set of all elements (l, w, m) in $\omega^{\omega} \times (\omega^{\omega})^{\omega} \times (\omega^{\omega \times \omega})^{\omega}$ that satisfy conditions 1–5 above. An element (l, w, m) of \mathcal{BD} codes the Bratteli diagram with $l(n)$ vertices at the n -th level of weight $w_n(0), \dots, w_n(l(n)-1)$ and with $m_n(i, j)$ arrows from the i -th vertex at the n -th level to the j -th vertex and the $(n+1)$ -st level for $n \in \omega$, $i \in l(n)$, and $j \in l(n+1)$. We call the elements of \mathcal{BD} simply “Bratteli diagrams”.

4.2. Dimension groups.

Definition 4.2.1. An *ordered abelian group* is a pair (G, G^+) , where G is an abelian group and G^+ is a subset of G satisfying

- (1) $G^+ + G^+ \subseteq G^+$;
- (2) $0 \in G^+$;
- (3) $G^+ \cap (-G^+) = \{0\}$;
- (4) $G^+ - G^+ = G$.

We call G^+ the *positive cone* of G . It defines an order on G by declaring that $x \leq y$ whenever $y - x \in G^+$. An element u of G^+ is said to be an *order unit* for (G, G^+) , if for every x in G , there exists a positive integer n such that

$$-nu \leq x \leq nu.$$

An ordered abelian group (G, G^+) is said to be *unperforated* if whenever a positive integer n and $a \in A$ satisfy $na \geq 0$, then $a \geq 0$. Equivalently, G^+ is divisible.

An ordered abelian group is said to have the *Riesz interpolation property* if for every $x_0, x_1, y_0, y_1 \in G$ such that $x_i \leq y_j$ for $i, j \in 2$, there is $z \in G$ such that

$$x_i \leq z \leq y_j$$

for $i, j \in 2$.

Definition 4.2.2. A *dimension group* is an unperforated ordered abelian group (G, G^+, u) with the Riesz interpolation property and a distinguished order unit u .

Let (G, G^+, u) and (H, H^+, w) be dimension groups, and let $\phi: G \rightarrow H$ be a group homomorphism.

- (1) We say that ϕ is *positive* if $\phi(G^+) \subseteq H^+$, and
- (2) We say that ϕ *preserves the unit* if $\phi(u) = w$.

Notice that positivity for a homomorphism between ordered groups is equivalent to preservation of the order.

Example 4.2.3. If $l \in \omega$ and $w_0, \dots, w_{l-1} \in \mathbb{N}$, then \mathbb{Z}^l with \mathbb{N}^l as its positive cone, and (w_0, \dots, w_{l-1}) as order unit, is a dimension group. We denote by $e_0^{(l)}, \dots, e_{l-1}^{(l)}$ the canonical basis of \mathbb{Z}^l .

We refer the reader to [51, Section 1.4] for a more complete exposition on dimension groups.

A dimension group can be coded in a natural way as an element of $\omega^{\omega \times \omega} \times 2^\omega \times \omega$. The set \mathcal{DG} of codes for dimension groups is a Borel subset of $\omega^{\omega \times \omega} \times 2^\omega \times \omega$, which can be regarded as the standard Borel space of dimension groups.

One can associate to a Bratteli diagram (l, w, m) the dimension group $G_{(l, w, m)}$ obtained as follows. For n in ω , denote by

$$\varphi_n: \mathbb{Z}^{l(n)} \rightarrow \mathbb{Z}^{l(n+1)}$$

the homomorphism given on the canonical bases of $\mathbb{Z}^{l(n)}$ by

$$\varphi_n \left(e_k^{(l(n))} \right) = \sum_{i \in l(n+1)} m_n(i, j) e_j^{(l(n+1))},$$

for all k in $l(n)$. Then $G_{(l, w, m)}$ is defined as the inductive limit of the inductive system

$$\left(\mathbb{Z}^{l(n)}, (w_n(0), \dots, w_n(l(n) - 1)), \varphi_n \right)_{n \in \omega}.$$

Theorem 2.2 in [14] asserts that any dimension group is in fact isomorphic to one of the form $G_{(l, w, m)}$ for some Bratteli diagram (l, w, m) . The key ingredient in the proof of [14, Theorem 2.2] is a Lemma due to Shen, see [14, Lemma 2.1] and also [55, Theorem 3.1]. We reproduce here the statement of the Lemma, for convenience of the reader.

Lemma 4.2.4. *Suppose that (G, G^+, u) is a dimension group. If $n \in \omega$ and $\theta: n \rightarrow G$ is any function, then there are $N \in \omega$, and functions $\Phi: N \rightarrow G^+$ and $g: n \times N \rightarrow \omega$, satisfying the following conditions:*

(1) *For all $i \in n$,*

$$\theta(i) = \sum_{j \in N} g(i, j) \Phi(j).$$

(2) *Whenever $(k_i)_{i \in n} \in \mathbb{Z}^n$ is such that $\sum_{i \in n} k_i \theta(i) = 0$, then*

$$\sum_{i \in l^G(n)} k_i g(i, j) = 0$$

for every $j \in N$.

It is immediate to note that the set of tuples $((G, G^+, u), n, \theta, N, \Phi, g)$ satisfying 1 and 2 of Lemma 4.2.4 is Borel. It follows that in Lemma 4.2.4 the number N and the maps Φ and g can be computed from (G, G^+, u) , n , and θ in a Borel way. This will be used to show that if we start with a dimension group (G, G^+, u) , the choice of the Bratteli diagram (l^G, w^G, m^G) satisfying $G_{l^G, w^G, m^G} \cong G$ as dimension groups with order units, which is guaranteed to be possible by [14, Theorem 2.2], can be made in a Borel way. See Proposition 4.2.5 below. A Borel version of [14, Theorem 2.2] is also proved in [16, Theorem 5.3]. We present here a proof, for the convenience of the reader, and to introduce ideas and notations to be used in the proof of Proposition 4.5.1.

Proposition 4.2.5. *There is a Borel function that associates to a dimension group $G = (G, G^+, u) \in \mathcal{DG}$ a Bratteli diagram $(l^G, w^G, m^G) \in \mathcal{BD}$ such that the dimension group associated with (l^G, w^G, m^G) is isomorphic to G .*

Proof. It is enough to construct in a Borel way a Bratteli diagram (l^G, w^G, m^G) and maps $\theta_n^G: l^G(n) \rightarrow G$ satisfying the following conditions:

(1) For all $i \in l^G(n)$,

$$\theta_n^G(i) = \sum_{j \in l^G(n+1)} m_n^G(i, j) \theta_n^G(j);$$

(2) For any $k_0, \dots, k_{l^G(n)-1} \in \mathbb{Z}$ such that

$$\sum_{i \in l^G(n)} k_i \theta_n^G(i) = 0,$$

we have that

$$\sum_{i \in l^G(n)} k_i m_n^G(i, j) = 0,$$

for every $j \in l^G(n+1)$;

(3) For every $x \in G^+$, there are $n \in \omega$ and $i \in l^G(n)$, such that $\theta_n^G(i) = x$.

It is not difficult to verify that conditions (1), (2) and (3) ensure that the dimension group coded by the Bratteli diagram (l^G, w^G, m^G) is isomorphic to G , via the isomorphism coded by $(\theta_n^G)_{n \in \omega}$.

We define θ_n^G , $l^G(n)$, w_n^G and m_n^G by recursion on n . Define $l^G(0) = 1$ and $\theta^G(0) = u$. Suppose that $l^G(k)$, w_k^G , m_k^G , and θ_k^G have been defined for $k \leq n$. Define $\theta': l^G(n) + 1 = \{0, \dots, l^G(n)\} \rightarrow G$ by

$$\theta'(i) = \begin{cases} \theta(i), & \text{if } i \in l^G(n), \\ n, & \text{if } i = l^G(n) \text{ and } n \in G^+, \\ u, & \text{otherwise.} \end{cases}$$

Suppose that the positive integer N and the functions $\Phi: N \rightarrow G^+$ and $g: n \times N \rightarrow \omega$ are obtained from $l^G(n)$ and θ' via Lemma 4.2.4. Define now:

$$\begin{aligned} l^G(n+1) &= N \\ m_n(i, j) &= \begin{cases} g(i, j), & \text{if } i \in l^G(n) \text{ and } j \in l^G(n+1), \\ 0, & \text{otherwise.} \end{cases} \\ w_{n+1}(j) &= \sum_{i \in l^G(n)} w_n(i, n) m_n(i, j). \end{aligned}$$

It is left as an exercise to check that with these choices, conditions (1), (2) and (3) are satisfied. This finishes the proof. \square

4.3. Approximately finite dimensional C*-algebras. A unital C*-algebra A is said to be *approximately finite dimensional*, or *AF-algebra* if it is isomorphic to a direct limit of a direct system of finite dimensional C*-algebras with unital connecting maps.

It is a standard result in the theory of C*-algebras, that any finite dimensional C*-algebra is isomorphic to a direct sum of matrix algebras over the complex numbers [12, Theorem III.1.1.]. A fundamental result due to Bratteli (building on previous work of Glimm) asserts that unital AF-algebras are

precisely the unital C^* -algebras that can be locally approximated by finite dimensional C^* -algebras.

Theorem 4.3.1 (Bratteli-Glimm [3, 23]). *Let A be a separable C^* -algebra. Then the following are equivalent:*

- (1) *A is a unital AF-algebra;*
- (2) *For every finite subset F of A and every $\varepsilon > 0$, there exists a finite dimensional C^* -subalgebra B of A , such that for every $a \in F$ there is $b \in B$ such that $\|a - b\| < \varepsilon$.*

A modern presentation of the proof of Theorem 4.3.1 can be found in [51, Proposition 1.2.2].

A distinguished class of unital AF-algebras are the so called unital *UHF-algebras*. These are the unital AF-algebras that are isomorphic to a direct limit of *full matrix algebras*. Of particular importance are the UHF-algebras *of infinite type*. These can be described as follows: Fix a strictly positive integer n . Denote by \mathbb{M}_n the C^* -algebra obtained as a limit of the inductive system

$$\mathbb{M}_n \rightarrow \mathbb{M}_{n^2} \rightarrow \mathbb{M}_{n^3} \rightarrow \dots$$

where the inclusion of \mathbb{M}_{n^k} into $\mathbb{M}_{n^{k+1}}$ is given by the diagonal embedding $a \mapsto \text{diag}(a, \dots, a)$. The UHF-algebras of infinite type are precisely those ones of the form \mathbb{M}_{n^∞} for some $n \in \mathbb{N}$.

A celebrated theorem of Elliott asserts that unital AF-algebras are classified up to isomorphism by their ordered K_0 -group. The K_0 -group is an ordered abelian group with a distinguished order unit that can be associated to any unital C^* -algebra, and bears information about the projections of the given C^* -algebra. The definition of the K_0 -group and its basic properties can be found in [50, Chapter 3]. We will denote by $K_0(A)$ the K_0 -group of the unital C^* -algebra A with positive cone $K_0(A)^+$ and distinguished order unit $[1_A]$ corresponding to the class of the unit of A . Dimension groups can be characterized within the class of ordered abelian groups with a distinguished order unit as the K_0 -groups of AF-algebras.

Theorem 4.3.2 (Elliott [15]). *Let A and B be unital AF-algebras.*

- (1) *For every positive morphism*

$$\phi: (K_0(A), K_0(A)^+) \rightarrow (K_0(B), K_0(B)^+),$$

such that $\phi([1_A]) \leq [1_B]$, there exists a homomorphism $\rho: A \rightarrow B$ such that $K_0(\rho) = \phi$.

- (2) *A and B are isomorphic if and only if $(K_0(A), K_0(A)^+, [1_A])$ and $(K_0(B), K_0(B)^+, [1_B])$ are isomorphic as dimension groups with order units.*

In (1), the range of ρ is a corner of B if and only if the set

$$\{x = [p] - [q] \in K_0(B)^+ : p, q \in B \text{ and } x \leq \phi([1_A])\}$$

is contained in $\phi(K_0(A)^+)$.

Let (l, w, m) be a Bratteli diagram. We will describe how to canonically associate to it a unital AF-algebra, which we will denote by $A_{(l, w, m)}$. For each n in ω , define a finite dimensional C*-algebra F_n by

$$F_n = \bigoplus_{i \in l(n)} \mathbb{M}_{w_n(i)}.$$

Denote by $\varphi_n: F_n \rightarrow F_{n+1}$ the unital injective *-homomorphism determined as follows. For every $i \in l(n)$ and $j \in l(n+1)$, the restriction of φ_n to the i -th direct summand of F_n and the j -th direct summand of F_{n+1} is a diagonal embedding of $m_n(i, j)$ copies of $\mathbb{M}_{w_n(i)}$ in $\mathbb{M}_{w_{n+1}(j)}$. Then $A_{(l, w, m)}$ is the inductive limit of the inductive system $(F_n, \varphi_n)_{n \in \omega}$.

The K_0 -group of $A_{(l, w, m)}$ is isomorphic to the dimension group $G_{l, w, m}$ associated with (l, w, m) .

The main result of [3] asserts that any unital AF-algebra is isomorphic to the C*-algebra associated with a Bratteli diagram. We show below that the code for such an AF-algebra can be computed in a Borel way.

Proposition 4.3.3. *Given a Bratteli diagram, there is a Borel way to compute the code for its associated unital AF-algebra.*

Proof. By Proposition 2.7.1, the inductive limit of an inductive system of C*-algebras can be computed in a Borel way. It is therefore enough to show that there is a Borel map that assigns to each Bratteli diagram, a code for the corresponding inductive system of C*-algebras. We will work, for convenience, with the parametrization $\Gamma(H)$ of C*-algebras.

Let $\{\xi_n^k: (k, n) \in \omega \times \omega\}$ be an orthonormal basis of H . For $n, m, k \in \omega$, denote by $e_{n, m}^{(k)}$ the rank 1 operator in $B(H)$ sending ξ_n^k to ξ_m^k . For convenience, we will also identify $\Gamma(H)$ with the space of nonzero functions from $\omega \times \omega \times \omega$ to the unit ball of $B(H)$. For $n \in \omega$, define $\gamma^{(n)} \in \Gamma(H)$ by

$$\gamma_{i, j, k}^{(n)} = \begin{cases} e_{i, j}^{(k)}, & \text{if } k \in l(n) \text{ and } i, j \in w_n(k), \\ 0, & \text{otherwise.} \end{cases}$$

Denote by $A_{i, j, k}^n$ the set of triples in $\omega \times \omega \times \omega$ of the form

$$\left(\sum_{k' \in k} w_n(k') m_n(k', t, n) + d w_n(k) + i, \sum_{k' \in k} w_n(k') m_n(k', t) + d w_n(n), t \right)$$

such that $d \in m_n(k, t)$, $i, j \in w_n(k)$ and $t \in l(n+1)$. It is clear that $C^*(\gamma^{(n)})$ is a finite dimensional C*-algebra isomorphic to

$$\bigoplus_{i \in l(n)} \mathbb{M}_{w(i, n)}.$$

For n in ω , let $\Phi^{(n)}: \mathcal{U} \rightarrow \mathcal{U}$ be the unique morphism of $\mathbb{Q}(i)$ -algebras satisfying

$$\Phi^{(n)}(X_{ijk}) = \sum_{(a, b, t) \in A_{ijk}^n} X_{a, b, t},$$

for i, j, k in ω . Then $\Phi^{(n)}$ is a code for the unital injective $*$ -homomorphism from $C^*(\gamma^{(n)})$ to $C^*(\gamma^{(n+1)})$ given by the diagonal embedding of $m_n(k, t, n)$ copies of $\mathbb{M}_{w_n(k)}$ in $\mathbb{M}_{w_n(t)}$ for every $k \in l(n)$ and $t \in l(n+1)$. By construction, the map $\mathcal{BD} \rightarrow R_{dir}(\Gamma(H))$ that assigns to every Bratteli diagram (l, w, m) , the code $(\gamma^{(n)}, \Phi^{(n)})_{n \in \omega}$, is Borel. This finishes the proof. \square

Proposition 4.2.5 together with Proposition 4.3.3 imply the following corollary.

Corollary 4.3.4. *There is a Borel map that assigns to a dimension group D a unital AF-algebra A_D whose ordered K_0 -group is isomorphic to D as dimension group.*

Since by [18, Proposition 3.4] the K_0 -group of a C^* -algebra can be computed in a Borel way, one can conclude that if \mathcal{A} is any Borel set of dimension groups, then the relation of isomorphisms restricted to \mathcal{A} is Borel bireducible with the relation of isomorphism unital AF-algebras whose K_0 -group is isomorphic to an element of \mathcal{A} .

Fix $n \in \mathbb{N}$. A dimension group has *rank* n if n is the largest size of a linearly independent subset. Let us denote by \cong_n^+ the relation of isomorphisms of dimension groups of rank n , and by \cong_n^{AF} the relation of isomorphisms of AF-algebras whose dimension group has rank n . By the previous discussion, the relations \cong_n^+ and \cong_n^{AF} are Borel bireducible. Moreover [16, Theorem 1.11] asserts that

$$\cong_n^+ <_B \cong_{n+1}^+$$

for every $n \in \mathbb{N}$. This means that \cong_n^+ is Borel reducible to \cong_{n+1}^+ , but \cong_{n+1}^+ is *not* Borel reducible to \cong_n^+ . It follows that the same conclusions hold for the relations \cong_n^{AF} : For every $n \in \mathbb{N}$

$$\cong_n^{AF} <_B \cong_{n+1}^{AF}.$$

This amounts at saying that it is strictly more difficult to classify AF-algebras with K_0 -group of rank $n+1$ than classifying AF-algebras with K_0 -group of rank n .

4.4. Enomorphisms of Bratteli diagrams.

Definition 4.4.1. Let $T = (l, w, m)$ be a Bratteli diagram. We say that an element $q = (q_n)_{n \in \omega} \in (\omega^{\omega \times \omega})^\omega$ is an *endomorphism* of T , if for every $n \in \omega$, $i \in l(n)$ and $t' \in l(n+1)$, the following identity holds

$$(1) \quad \sum_{t \in l(n+1)} m_n(i, t) q_{n+1}(t, t') = \sum_{t \in l(n+1)} q_n(i, t) m_{n+1}(t, t').$$

The set $End_{\mathcal{BD}}$ of pairs $(T, q) \in \mathcal{BD} \times (\omega^{\omega \times \omega})^\omega$ such that T is a Bratteli diagram and q is an endomorphism of T , is Borel.

We proceed to describe how an endomorphism of a Bratteli diagram, in the sense of the definition above, gives rise to an endomorphism of the unital AF-algebra associated with it. Let $(F_n, \varphi_n)_{n \in \omega}$ be the inductive system of

finite dimensional C^* -algebras associated with T , and denote by A_T its inductive limit. By repeatedly applying [12, Lemma III.2.1], one can define unital $*$ -homomorphisms $\psi_n: F_n \rightarrow F_{n+1}$ for n in ω , satisfying the following conditions:

- (1) ψ_n is unitarily equivalent to the $*$ -homomorphism from F_n to F_{n+1} such that for every $i \in l(n)$ and $j \in l(n+1)$ the restriction of ψ_n to the i -th direct summand of F_n and the j -th direct summand of F_{n+1} is a diagonal embedding of $q_n(i, j)$ copies of $\mathbb{M}_{w_n(i)}$ in $\mathbb{M}_{w_{n+1}(j)}$;
- (2) $\psi_n \circ \varphi_{n-1} = \varphi_n \circ \psi_{n-1}$ whenever $n \geq 1$.

(Notice in particular that ψ_0 is determined solely by condition (1).) One thus obtains a one sided intertwining $(\psi_n)_{n \in \omega}$ from $(F_n, \varphi_n)_{n \in \omega}$ to itself. We denote by $\psi_{T,q}: A_T \rightarrow A_T$ the corresponding inductive limit endomorphism.

Proposition 4.4.2. *Given a Bratteli diagram T and an endomorphism q of T , there is a Borel way to compute a code for the endomorphism $\psi_{T,q}$ of A_T associated with q .*

Proof. By Proposition 2.8.2, a code for the limit homomorphism between two inductive limits of C^* -algebras can be computed in a Borel way. It is therefore enough to show that there is a Borel function from $End_{\mathcal{BD}}$ to $R_{int}(\Gamma(H))$ assigning to an element (T, q) of $End_{\mathcal{BD}}$, a code for the corresponding one sided intertwining system.

Let $((l, w, m), q)$ be an element in $End_{\mathcal{BD}}$, and let $\{\xi_m^n: (n, m) \in \omega \times \omega\}$ be an orthonormal basis of H . Denote by $(\gamma^{(n)}, \Phi^{(n)})_{n \in \omega}$ the element of $R_{dir}(\Gamma(H))$ associated with the Bratteli diagram (l, w, m) as in the proof of Proposition 4.3.3. For n in ω , define

$$u^{(n)} = \sum_{k \in l(n)} \sum_{i \in w_n(k)} e_{ii}^{(k)},$$

which is an element of $B(H)$. Observe that $u^{(n)}$ is the unit of $C^*(\gamma^{(n)})$. We define the sequence $(\Psi^{(n)})_{n \in \omega}$ in $(\mathcal{U}^{\mathcal{U}})^{\omega}$ as follows. Let $A_{i,j,k}^n$ denote the set of triples

$$\left(\sum_{k' \in k} w_n(k') q_n(k', t, n) + dw_n(k) + i, \sum_{k' \in k} w_n(k') q_n(k', t) + dw_n(k), t \right)$$

such that d belongs to $q_n(k, t)$, i and j belong to $w_n(k)$, and t belongs to $l(n+1)$. Let $\psi^{(n)}: \mathcal{U} \rightarrow \mathcal{U}$ be the unique homomorphism of $\mathbb{Q}(i)$ -algebras satisfying

$$\psi^{(n)}(X_{ijk}) = \sum_{(a,b,t) \in A_{ijk}^n} X_{a,b,t}$$

for i, j and k in ω . For $p \in \mathcal{U}$, set

$$\Psi_0^{(n)}(p) = \psi^{(n)}(p).$$

By construction, the elements

$$\left(\Psi_0^{(n)} \circ \Phi_k^{(n-1)}\right)(p) \left(\gamma^{(n-1)}\right) \quad \text{and} \quad \left(\Phi_k \circ \Psi_0^{(n-1)}\right)(p) \left(\gamma^{(n-1)}\right)$$

are unitarily equivalent in $C^*(\gamma^{(n+1)})$ for every $k \in \omega$. Using that $\gamma^{(n)}$ is a unitary for all n in ω , choose elements $p_k^{(n)}$ in \mathcal{U} , for k in ω , satisfying the following conditions:

$$\begin{aligned} \left\| p_k^{(n)} \left(\gamma^{(n+1)}\right) p_k^{(n)} \left(\gamma^{(n+1)}\right)^* - 1 \right\| &< \frac{1}{k+1} \\ \left\| p_k^{(n)} \left(\gamma^{(n+1)}\right)^* p_k^{(n)} \left(\gamma^{(n+1)}\right) - 1 \right\| &< \frac{1}{k+1} \\ \left\| p_k^{(n)} \left(\gamma^{(n+1)}\right) - p_m^{(n)} \left(\gamma^{(n+1)}\right) \right\| &< \frac{1}{\min\{k, m\} + 1} \end{aligned}$$

and

$$\begin{aligned} \left\| p_k^{(n)} \left(\gamma^{(n+1)}\right) \left(\left(\Psi_0^n \circ \Phi_k^{n-1}\right)(p) \left(\gamma^{(n-1)}\right) \right) p_k^{(n)} \left(\gamma^{(n+1)}\right)^* \right. \\ \left. - \left(\Phi_k \circ \Psi_0^{(n-1)}\right)(p) \left(\gamma^{(n-1)}\right) \right\| &< \frac{1}{k+1}. \end{aligned}$$

Finally, define

$$\Psi_k^{(n)}(p) = p_k^{(n)} \Psi_0^{(n)}(p) \left(p_k^{(n)}\right)^*$$

for all p in \mathcal{U} . It is clear that for fixed n in ω , the sequence $\Psi^{(n)} = (\Psi_k^{(n)})_{k \in \omega}$ is a code for a *-homomorphism $\widehat{\psi^{(n)}}: \widehat{\gamma^{(n)}} \rightarrow \widehat{\gamma^{(n+1)}}$ that moreover satisfies

$$\widehat{\Psi}^{(n)} \circ \widehat{\Phi}^{(n-1)} = \widehat{\Phi}^{(n)} \circ \widehat{\Psi}^{(n-1)}.$$

Thus,

$$\left(\left(\gamma^{(n)}, \Phi^{(n)}\right)_{n \in \omega}, \left(\gamma^{(n)}, \Phi^{(n)}\right)_{n \in \omega}, \left(\Psi^{(n)}\right)_{n \in \omega} \right)$$

is an element in $R_{int}(\Gamma(H))$. It is clear that this is a code for the one sided intertwining system associated with $((l, m, w), q)$, and that it can be computed in a Borel fashion. \square

4.5. Endomorphisms of dimension groups. Let (G, G^+, u) be a dimension group. Let us denote by $End_{\mathcal{DG}}$ the set of pairs $(G, \phi) \in \mathcal{DG} \times \omega^\omega$ such that G is a dimension group and ϕ is an endomorphism of G .

Let (l, w, m) be a Bratteli diagram, and let

$$\left(\mathbb{Z}^{l(n)}, (w_n(0), \dots, w_n(l(n)-1)), \varphi_n \right)_{n \in \omega}$$

be the inductive system of dimension groups whose inductive limit is the dimension group $G_{l,w,m}$ associated with (l, w, m) . Fix an endomorphism q

of (l, w, m) , and for $n \in \omega$, define a positive homomorphism $\psi_n: \mathbb{Z}^{l(n)} \rightarrow \mathbb{Z}^{l(n+1)}$ by

$$\psi_n(e_i^{(l(n))}) = \sum_{j \in l(n+1)} q_n(i, j) e_j^{(l(n+1))}.$$

Observe that the sequence $(\psi_n)_{n \in \omega}$ induces an inductive limit endomorphism $\phi_{((l, w, m), q)}$ of $G_{(l, w, m)}$.

Proposition 4.5.1. *There is a Borel map*

$$\begin{aligned} \text{End}_{\mathcal{DG}} &\rightarrow \text{End}_{\mathcal{BD}} \\ (G, \phi) &\mapsto (T^G, q^{G, \phi}), \end{aligned}$$

such that the dimension group associated with T^G is isomorphic to G , and the endomorphism of the dimension group associated with T^G corresponding to $q^{G, \phi}$, is conjugate to ϕ .

Proof. It is enough to construct, in a Borel way,

- a Bratteli diagram $(l^{G, \phi}, w^{G, \phi}, m^{G, \phi})$,
- an endomorphism $q^{G, \phi}$ of $(l^{G, \phi}, w^{G, \phi}, m^{G, \phi})$, and
- functions $\theta_n^{G, \phi}: l^{G, \phi}(n) \rightarrow G^+$ for n in ω ,

such that the following conditions hold:

- (1) For every $i \in l^{G, \phi}(n)$,

$$\theta_n^{G, \phi}(i) = \sum_{j \in l^{G, \phi}(n+1)} m_n^{G, \phi}(i, j) \theta_n^{G, \phi}(j);$$

- (2) For any $k_0, \dots, k_{l(n)-1} \in \mathbb{Z}$ such that

$$\sum_{i \in l^{G, \phi}(n)} k_i \theta_n^{G, \phi}(i) = 0,$$

we have that

$$\sum_{i \in l^{G, \phi}(n)} k_i m_n^{G, \phi}(i, j) = 0,$$

for every $j \in l^{G, \phi}(n+1)$;

- (3) For every $x \in G^+$, there are $n \in \omega$ and $i \in l^{G, \phi}(n)$ such that $\theta_n^{G, \phi}(i) = x$;

- (4) $\phi(\theta_n^{G, \phi}(i)) = q^G(i)$ for $n \in \omega$ and $i \in l^{G, \phi}(n)$.

In fact, it is not difficult to see that Conditions (1), (2), (3), and (4) ensure that $(\theta_n^{G, \phi})_{n \in \omega}$ defines an isomorphism from the dimension group associated with $(l^{G, \phi}, w^{G, \phi}, m^{G, \phi})$ to G that conjugates the endomorphism associated with $q^{G, \phi}$ and ϕ .

We define $l_n^{G,\phi}, w_n^{G,\phi}, m_n^{G,\phi}, q_n^{G,\phi}$ and $\theta_n^{G,\phi}$ satisfying conditions 1–4 by recursion on n . Define $l^{G,\phi}(0) = 1$ and $\theta_0^{G,\phi}(0) = u$. Suppose that $l^{G,\phi}(k), w_k^{G,\phi}, m_{k-1}^{G,\phi}$, and $\theta_k^{G,\phi}$ have been defined for $k \leq n$. Define

$$\theta': 2l^{G,\phi}(n) + 1 = \{0, \dots, 2l^{G,\phi}(n)\} \rightarrow \omega$$

by

$$\theta'(i) = \begin{cases} \theta^{G,\phi}(i) & \text{if } 0 \leq i < l^{G,\phi}(n), \\ \phi(\theta^{G,\phi}(i - l^{G,\phi}(n))) & \text{if } l^{G,\phi}(n) \leq i < 2l^{G,\phi}(n), \\ n & \text{if } i = 2l^{G,\phi}(n) \text{ and } n \in G^+, \\ u & \text{otherwise.} \end{cases}$$

Suppose that the positive integer N , and the functions $\Phi: N \rightarrow G^+$ and $g: (2l^{G,\phi}(n) + 1) \times N \rightarrow \omega$ are obtained via Lemma 4.2.4 from $2l^{G,\phi}(n) + 1$ and θ' , and let $N' \in \omega$, $\Phi': N' \rightarrow \omega$, and $g': N \times N' \rightarrow \omega$ satisfy the conclusion of Lemma 4.2.4 for the choices N and Φ . Define now:

$$\begin{aligned} l^{G,\phi}(n+1) &= N'; \\ w_n^{G,\phi}(j) &= \sum_{i \in l^{G,\phi}(n)} w_n^{G,\phi}(i) m_n^{G,\phi}(i, j) \\ m_n^{G,\phi}(i, j) &= \begin{cases} \sum_{t \in N} g(i, t) g'(t, j) & \text{if } i \in l^{G,\phi}(n) \text{ and } j \in l^{G,\phi}(n+1), \\ 0 & \text{otherwise.} \end{cases} \\ q_n^{G,\phi}(i, j) &= \sum_{t \in N} g(2i, t) g'(t, j) \\ \theta_{n+1}^{G,\phi} &= \Phi. \end{aligned}$$

It is not difficult to check that this recursive construction gives maps satisfying conditions 1–4. \square

Corollary 4.5.2. *There is a Borel map that assigns to a dimension group G with a distinguished endomorphism ϕ , a code for a unital AF-algebra A and a code for an endomorphism ρ of A , such that the K_0 -group of A is isomorphic to G as dimension groups with order units, and the endomorphism of the K_0 -group of A corresponding to ρ is conjugate to ϕ .*

Proof. Let G be a dimension group, and let ϕ be an endomorphism of G . Using Proposition 4.5.1, choose in a Borel way a Bratteli diagram (l, m, w) and an endomorphism q of (l, m, w) such that G is isomorphic to the dimension group associated with (l, m, w) , and ρ is conjugate to the endomorphism associated with q . Use Proposition 4.3.3 to choose in a Borel way, a unital AF-algebra A whose Bratteli diagram is (l, m, w) . Apply Proposition 4.4.2 to choose in a Borel way an endomorphism ρ of A whose induced endomorphism of the Bratteli diagram is q . It is clear from the construction that

the K_0 -group of A is isomorphic to G . Moreover, ϕ is conjugate to the endomorphism of the K_0 -group of A corresponding to ρ . Therefore, the result follows from Proposition 4.5.1 and Proposition 4.4.2. \square

5. COCYCLE CONJUGACY OF AUTOMORPHISMS OF \mathcal{O}_2

5.1. Strongly self-absorbing C^* -algebras. Upon studying the literature around Elliott's classification program, it is clear that certain C^* -algebras play a central role in major stages of the program: UHF-algebras (particularly those of infinite type), the Cuntz algebras \mathcal{O}_2 and \mathcal{O}_∞ [10], and, more recently, the Jiang-Su algebra \mathcal{Z} [28]. In [58], Toms and Winter were able to abstract the property which singles these algebras out among other similar C^* -algebras. The relevant notion is that of strongly self-absorbing C^* -algebras, which we define below; see also [58, Definition 1.3].

Definition 5.1.1. Let \mathcal{D} be a separable, unital, infinite dimensional C^* -algebra. Denote by $\mathcal{D} \otimes \mathcal{D}$ the completion of the algebraic tensor product $\mathcal{D} \odot \mathcal{D}$ with respect to any compatible C^* -norm on $\mathcal{D} \odot \mathcal{D}$. We say that \mathcal{D} is *strongly self-absorbing* if there exists an isomorphism $\varphi: \mathcal{D} \rightarrow \mathcal{D} \otimes \mathcal{D}$ which is approximately unitarily equivalent to the map $a \mapsto a \otimes 1_{\mathcal{D}}$.

It is shown in [58] that a C^* -algebra \mathcal{D} satisfying Definition 5.1.1 is automatically nuclear. In particular the choice of the tensor product norm on $\mathcal{D} \odot \mathcal{D}$ is irrelevant. By [58, Examples 1.14] the following C^* -algebras are strongly-self-absorbing: UHF-algebras of infinite type, the Cuntz algebras \mathcal{O}_2 and \mathcal{O}_∞ , the tensor product of a UHF-algebra of infinite type and \mathcal{O}_∞ , and the Jiang-Su algebra. No other strongly self-absorbing C^* -algebra is currently known.

Definition 5.1.2. Suppose that D is a nuclear C^* -algebra. A C^* -algebra A absorbs D tensorially –or is *D -absorbing*– if the tensor product $A \otimes D$ is isomorphic to A .

The particular case of Theorem 5.1.3 when \mathcal{D} is the Jiang-Su algebra \mathcal{Z} has been proved in [18, Theorem A.1].

Theorem 5.1.3. *Suppose that \mathcal{D} is a strongly self-absorbing C^* -algebra. The set of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a \mathcal{D} -absorbing unital C^* -algebra is Borel.*

Proof. By [19, Lemma 3.14], the set $\Gamma_u(H)$ of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is unital, is Borel. Moreover, there is a Borel function $Un: \Gamma_u(H) \rightarrow B(H)$ such that $Un(\gamma)$ is the unit of $C^*(\gamma)$ for every $\gamma \in \Gamma_u(H)$. Let $\{d_n: n \in \omega\}$ be an enumeration of a dense subset of \mathcal{D} such that $d_0 = 1$, and let $\{p_n: n \in \omega\}$ be an enumeration of \mathcal{U} . By [58, Theorem 2.2], or [51, Theorem 7.2.2], a unital C^* -algebra A is \mathcal{D} -absorbing if and only if for every $n, m \in \mathbb{N}$ and every finite subset F of A , there are $a_0, a_1, \dots, a_n \in A$ such that

- a_0 is the unit of A ,

- $\|xa_i - a_ix\| < \frac{1}{m}$ for every $i \in n$ and $x \in F$, and
- $\|p_i(a_0, \dots, a_n) - p_i(d_0, \dots, d_n)\| < \frac{1}{m}$ for every $i \in m$.

Let $\gamma \in \Gamma(H)$ be such that $C^*(\gamma)$ is unital. Then $C^*(\gamma)$ is \mathcal{D} -absorbing if and only if for every $n, m \in \mathbb{N}$ there are $k_1, \dots, k_n \in \omega$ such that

- $\|\gamma_i \gamma_{k_j} - \gamma_{k_j} \gamma_i\| < \frac{1}{m}$ for $i \in m$ and $1 \leq j \leq n$,
- $\|p_i(Un(\gamma), \gamma_{k_1}, \dots, \gamma_{k_n}) - p_i(d_0, \dots, d_n)\| < \frac{1}{m}$ for every $i \in m$.

This shows that the set of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is unital and \mathcal{D} -absorbing, is Borel. \square

5.2. Background on Kirchberg algebras and the UCT. A simple C^* -algebra A is called *purely infinite*, if for every a and b in A with $a \neq 0$, there are x and y in A such that $xay = b$ [1, V.2.3.3]. For a unital C^* -algebra A this is equivalent to the assertion that for every a in A with $a \neq 0$, there are x and y in A such that $xay = 1$.

A *Kirchberg algebra* is a simple, separable, nuclear, purely infinite C^* -algebra [51, Chapter 4].

The following result of Kirchberg, which is [35, Theorem 3.15], is crucial in the study of purely infinite simple C^* -algebras.

Theorem 5.2.1 (Kirchberg, [35]). *If A is a nuclear simple C^* -algebra, then A is purely infinite if and only if A is \mathcal{O}_∞ -absorbing.*

We proceed to define what it means for a C^* -algebra to satisfy the Universal Coefficient Theorem, or UCT for short. In addition to the K_0 group, one can associate to a C^* -algebra another abelian group, namely the K_1 -group. Its definition and its basic properties can be found in [50, Chapter 8]. (We point out that unlike the K_0 -group, the K_1 -group of a C^* -algebra does not carry any natural ordering or positive cone.)

If A is a C^* -algebra, we denote by $K_*(A)$ the \mathbb{Z}_2 -graded abelian group $K_*(A) = K_0(A) \oplus K_1(A)$. If A and B are C^* -algebras, a group homomorphism $\nu: K_*(A) \rightarrow K_*(B)$ is said to have *degree zero* if $\nu(K_j(A)) \subseteq K_j(B)$ for $j = 0, 1$, and it is said to have *degree one* if $\nu(K_j(A)) \subseteq K_{1-j}(B)$ for $j = 0, 1$.

Definition 5.2.2. Let A and B be separable C^* -algebras. We say that the pair (A, B) *satisfies the UCT* if the following conditions are satisfied:

- (1) The natural degree zero map $\tau_{A,B}: KK(A, B) \rightarrow \text{Hom}(K_*(A), K_*(B))$ defined in [30], is surjective.
- (2) The natural degree one map $\mu_{A,B}: \ker(\tau_{A,B}) \rightarrow \text{Ext}(K_*(A), K_{*+1}(B))$ defined in [30], is an isomorphism.

If this is the case, by setting $\varepsilon_{A,B} = \mu_{A,B}^{-1}: \text{Ext}(K_*(A), K_{*+1}(B)) \rightarrow KK(A, B)$, we obtain a short exact sequence

$$0 \longrightarrow \text{Ext}(K_*(A), K_{*+1}(B)) \xrightarrow{\varepsilon_{A,B}} KK(A, B) \xrightarrow{\tau_{A,B}} \text{Hom}(K_*(A), K_*(B)) \longrightarrow 0,$$

which is natural on both variables because so are $\tau_{A,B}$ and $\mu_{A,B}$.

We further say that A *satisfies the UCT*, if (A, B) satisfies the UCT for every separable C^* -algebra B .

We say that two separable nuclear C^* -algebras A and B are *KK -equivalent* if there exist $x \in KK(A, B)$ and $y \in KK(B, A)$ such that $xy = 1_B$ and $yx = 1_A$.

The class of all separable nuclear C^* -algebras that satisfy the UCT, usually called the *bootstrap* class \mathcal{N} , can be characterized as the smallest class of separable nuclear C^* -algebras containing the complex numbers \mathbb{C} , and closed under the following operations:

- Countable direct sums,
- Crossed products by \mathbb{Z} and by \mathbb{R} ,
- Two out of three in extensions,
- KK -equivalence.

It can be shown that a C^* -algebra belongs to \mathcal{N} if and only if it is KK -equivalent to a commutative C^* -algebra. It is a long standing open problem whether all separable nuclear C^* -algebras belong to \mathcal{N} .

The UCT plays a key role in the classification of C^* -algebras, and it is a standard assumption in most classification theorems thus far available. (In many situations, as is the case for AF-algebras, the UCT is automatically satisfied.) We recall here Kirchberg-Phillips classification theorem for (unital) Kirchberg algebras satisfying the UCT.

Theorem 5.2.3 (Kirchberg, Phillips [34, 45]). *Let A and B be unital Kirchberg algebras satisfying the UCT. Then A and B are isomorphic if and only if there is a degree zero group isomorphism*

$$\varphi_*: K_*(A) \rightarrow K_*(B)$$

such that $\varphi_0([1_A]) = [1_B]$.

We point out that if A is a Kirchberg algebra, then the order structure on $K_0(A)$ is trivial, this is, $K_0(A)^+ = K_0(A)$. Thus, every element in $K_0(A)$ is positive.

5.3. Borel spaces of Kirchberg algebras. We will denote by $\Gamma_{uKir}(H)$ the set of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a *unital* Kirchberg algebra.

Proposition 5.3.1. *The set $\Gamma_{uKir}(H)$ is Borel.*

Proof. Corollary 7.5 of [19] asserts that the set $\Gamma_{uns}(H)$ of $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is unital, nuclear, and simple is Borel. The result then follows from this fact together with Theorem 5.1.3. \square

Definition 5.3.2. Fix a projection p in \mathcal{O}_∞ such that $[p] = 0$ in $K_0(\mathcal{O}_\infty) \cong \mathbb{Z}$. Define the *standard Cuntz algebra* \mathcal{O}_∞^{st} to be the corner $p\mathcal{O}_\infty p$.

Remark 5.3.3. The C^* -algebra \mathcal{O}_∞^{st} is a unital Kirchberg algebra that satisfies the UCT, with K -theory given by

$$(K_0(\mathcal{O}_\infty^{st}), [1_{\mathcal{O}_\infty^{st}}], K_1(\mathcal{O}_\infty^{st})) \cong (\mathbb{Z}, 0, 0).$$

Moreover, it is the unique, up to isomorphism, unital Kirchberg algebra satisfying the UCT with said K -theory. In particular, a different choice of the projection p in Definition 5.3.2 (as long as its class on K -theory is 0), would yield an isomorphic C^* -algebra.

We point out that, even though there is an isomorphism $\mathcal{O}_\infty^{st} \otimes \mathcal{O}_\infty^{st} \cong \mathcal{O}_\infty^{st}$ (see comments on page 262 of [26]), the C^* -algebra \mathcal{O}_∞^{st} is not strongly self-absorbing. Indeed, if \mathcal{D} is a strongly self-absorbing C^* -algebra, then the infinite tensor product $\bigotimes_{n=1}^\infty \mathcal{D}$ of \mathcal{D} with itself, is isomorphic to \mathcal{D} . However, $\bigotimes_{n=1}^\infty \mathcal{O}_\infty^{st}$ is isomorphic to \mathcal{O}_2 , and thus \mathcal{O}_∞^{st} is not strongly self-absorbing.

We proceed to give a K -theoretic characterization of those unital Kirchberg algebras that absorb \mathcal{O}_∞^{st} . Our characterization will be used to show that the set of all \mathcal{O}_∞^{st} -absorbing unital Kirchberg algebras is Borel.

For use in the proof of the following lemma, we recall here that if A and B are nuclear separable C^* -algebras, and at least one of them satisfies the UCT, then the K -groups of their tensor product $A \otimes B$ are “essentially” determined by the K -groups of A and B , up to an extension problem. This is the content of the Künneth formula, which will be needed in the next proof. We refer the reader to [53] for the precise statement and its proof; see also Remark 7.11 in [52].

Lemma 5.3.4. *Let A be a unital Kirchberg algebra. Then the following are equivalent:*

- (1) A is \mathcal{O}_∞^{st} -absorbing.
- (2) The class $[1_A]$ of the unit of A in $K_0(A)$ is zero.

Proof. We first show that (1) implies (2). Since \mathcal{O}_∞^{st} satisfies the UCT, the Künneth formula applied to $A \otimes \mathcal{O}_\infty^{st}$ gives

$$K_0(A \otimes \mathcal{O}_\infty^{st}) \cong K_0(A) \quad \text{and} \quad K_1(A \otimes \mathcal{O}_\infty^{st}) \cong K_1(A),$$

with $[1_{A \otimes \mathcal{O}_\infty^{st}}] = 0$ as an element in $K_0(A)$. The claim follows since any isomorphism $A \otimes \mathcal{O}_\infty^{st} \cong A$ must map the unit of $A \otimes \mathcal{O}_\infty^{st}$ to the unit of A .

Let us now show that (2) implies (1). Fix a non-zero projection p in \mathcal{O}_∞ such that $[p] = 0$ as an element of $K_0(\mathcal{O}_\infty)$. Then $1_A \otimes 1_{\mathcal{O}_\infty}$, which is an element of $A \otimes \mathcal{O}_\infty$, represents the zero group element in $K_0(A \otimes \mathcal{O}_\infty)$. Likewise, $1_A \otimes p$ also represents the zero group element in $K_0(A \otimes \mathcal{O}_\infty)$. Since any two non-zero projections in a Kirchberg algebra are Murray-von Neumann equivalent if and only if they determine the same class in K -theory (see [11]), it follows that there is an isometry v in $A \otimes \mathcal{O}_\infty$ such that

$$vv^* = 1_A \otimes p.$$

Define a map

$$\varphi_0: A \otimes_{alg} \mathcal{O}_\infty \rightarrow (1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p) \cong A \otimes \mathcal{O}_\infty^{st}$$

by $\varphi_0(a \otimes b) = v(a \otimes b)v^*$, for a in A and b in \mathcal{O}_∞ , and extended linearly. It is straightforward to check that φ_0 extends to a $*$ -homomorphism $\varphi: A \otimes \mathcal{O}_\infty \rightarrow A \otimes \mathcal{O}_\infty^{st}$. We claim that φ is an isomorphism. For this, it is enough to check that the $*$ -homomorphism

$$\psi: (1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p) \rightarrow A \otimes \mathcal{O}_\infty$$

given by $\psi(x) = v^*xv$ for all x in $(1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p)$, is an inverse for φ . This is immediate since $(1_A \otimes p)x(1_A \otimes p) = x$ for all x in $(1_A \otimes p)(A \otimes \mathcal{O}_\infty)(1_A \otimes p)$.

Once we have $A \otimes \mathcal{O}_\infty^{st} \cong A \otimes \mathcal{O}_\infty$, the result follows from the fact that there is an isomorphism $A \cong A \otimes \mathcal{O}_\infty$ by Kirchberg's \mathcal{O}_∞ -isomorphism Theorem (here reproduced as Theorem 5.2.1). This finishes the proof of the lemma. \square

Corollary 5.3.5. *The set of all $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a \mathcal{O}_∞^{st} -absorbing unital Kirchberg algebra, is Borel.*

Proof. This follows from Lemma 5.3.4, together with the fact that the K -theory of a C^* -algebra and the class of its unit in K_0 can be computed in a Borel fashion; see [18, Section 3.3]. \square

It follows from Theorem 5.1.3, together with the fact that a the UHF-algebra of infinite type \mathbb{M}_{n^∞} is strongly self-absorbing, that the set of all $\gamma \in \Gamma(H)$ such that $C^*(\gamma)$ is a \mathbb{M}_{n^∞} -absorbing unital Kirchberg algebra is Borel.

5.4. Isomorphism of p -divisible torsion free abelian groups.

Definition 5.4.1. Let G be an abelian group and let n be a positive integer.

- (1) We say that G is *n -divisible*, if for every x in G there exists y in G such that $x = ny$.
- (2) We say that G is *uniquely n -divisible*, if for every x in G there exists a unique y in G such that $x = ny$.

Given a set S of positive integers, we say that G is (uniquely) S -divisible, if G is (uniquely) n -divisible for every n in S .

It is clear that if n is a positive integer, then any n -divisible torsion free abelian group is uniquely n -divisible.

It is easily checked that the following classes of abelian groups are Borel subsets of the standard Borel space of groups \mathcal{G} :

- Torsion free groups;
- n -divisible groups, for any positive integer n ;
- Uniquely n -divisible groups.

The main result of [25] asserts that if \mathcal{C} is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphisms of elements of \mathcal{C} is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation \cong_{TFA} of isomorphism of torsion free abelian groups. Moreover, [13, Theorem 1.1] asserts that \cong_{TFA} is a complete analytic set and, in particular, not Borel.

Proposition 5.4.2. *Suppose that \mathcal{P} is a set of prime numbers which is coinfinite in the set of all primes. If \mathcal{C} is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of \mathcal{C} is Borel, then \mathcal{C} is Borel reducible to the relation of isomorphism of torsion free \mathcal{P} -divisible groups. Moreover, the latter equivalence relation is a complete analytic set and, in particular, not Borel.*

Proof. A variant of the argument used in the proof of the main result of [25] can be used to prove the first assertion. Indeed, the only modification needed is in the definition of the *group eplag* associated with an *excellent prime labeled graph* as in [25, Section 2] (we refer to [25] for the definitions of these notions). Suppose that (V, E, f) is an excellent prime labeled graph such that the range of f is disjoint from \mathcal{P} . Denote by $\mathbb{Q}^{(V)}$ the direct sum

$$\mathbb{Q}^{(V)} = \bigoplus_{v \in V} \mathbb{Q}$$

of copies of \mathbb{Q} indexed by V , and identify an element v of V with the corresponding copy of \mathbb{Q} in $\mathbb{Q}^{(V)}$. We define the \mathcal{P} -divisible group eplag $\mathcal{G}_{\mathcal{P}}(V, E, f)$ associated with (V, E, f) , to be the subgroup of $\mathbb{Q}^{(V)}$ generated by

$$\left\{ \frac{v}{p^n f(v)^m}, \frac{v+w}{p^n f(\{v, w\})} : v \in V, \{v, w\} \in E, n, m \in \omega, p \in \mathcal{P} \right\}.$$

It is easy to check that $\mathcal{G}_{\mathcal{P}}(V, E, f)$ is indeed a torsion-free \mathcal{P} -divisible abelian group. The group eplag $\mathcal{G}(V, E, f)$ as defined in [25, Section 2], is the particular case of this definition with $\mathcal{P} = \emptyset$. The same argument as in [25], where

- (1) the group eplag $\mathcal{G}(V, E, f)$ is replaced everywhere by $\mathcal{G}_{\mathcal{P}}(V, E, f)$, and
- (2) all the primes are chosen from the *complement* of \mathcal{P} ,

gives a proof of the first claim of this proposition.

The second claim follows by modifying the argument in [13] and, in particular, the construction of the torsion-free abelian group associated with a tree on ω as in [13, Theorem 2.1]. Choose injective enumerations $(p_n)_{n \in \omega}$ and $(q_n)_{n \in \omega}$ of disjoint subsets of the complement of \mathcal{P} in the set of all primes, and let T be a tree on ω . Define the excellent prime labeled graph (V_T, E_T, f_T) as follows. The graph (V_T, E_T) is just the tree T , and

$$f : V_T \cup E_T \rightarrow \{p_n, q_n : n \in \omega\}$$

is defined by

$$f(x) = \begin{cases} q_n & \text{if } x \text{ is a vertex in the } n\text{-th level of } T; \\ p_n & \text{if } x \text{ is an edge between the } n\text{-th and } n+1\text{-th levels of } T. \end{cases}$$

Define the \mathcal{P} -divisible torsion free abelian group $G_{\mathcal{P}}(T)$ to be the group eplag $\mathcal{G}_{\mathcal{P}}(V_T, E_T, f_T)$ as defined in the first part of the proof. The same proof as that of [13, Theorem 2.1] shows the following facts: If T and T' are isomorphic trees, then the groups $G_{\mathcal{P}}(T)$ and $G_{\mathcal{P}}(T')$ are isomorphic. On the other hand, if T is well-founded and T' is ill-founded, then $G_{\mathcal{P}}(T)$ and $G_{\mathcal{P}}(T')$ are not isomorphic. The second claim of this proposition can now be proved as [13, Theorem 1.1]. \square

5.5. Automorphisms of \mathcal{O}_2 . Denote by $\text{Aut}(\mathcal{O}_2)$ the Polish group of automorphisms of \mathcal{O}_2 with respect to the topology of pointwise convergence. Given a positive integer n , the closed subspace $\text{Aut}_n(\mathcal{O}_2)$, of automorphisms of \mathcal{O}_2 of order n can be identified with the space of actions of \mathbb{Z}_n on \mathcal{O}_2 .

Definition 5.5.1. (See [26, Definition 3.6]) An automorphism α in $\text{Aut}_n(\mathcal{O}_2)$ is said to be *approximately representable* if for every $\varepsilon > 0$ and for every finite subset F of \mathcal{O}_2 , there exists a unitary u of \mathcal{O}_2 such that

- (1) $\|u^n - 1\| < \varepsilon$,
- (2) $\|\alpha(u) - u\| < \varepsilon$, and
- (3) $\|\alpha(a) - uau^*\| < \varepsilon$ for every $a \in F$.

It is clear that the set of approximately representable automorphisms of order n of \mathcal{O}_2 is a G_δ subset of $\text{Aut}_n(\mathcal{O}_2)$.

For a prime number p , we introduce below a certain model action of \mathbb{Z}_p on \mathcal{O}_2 . When $p = 2$, this is the example constructed by Izumi on page 262 of [26].

Definition 5.5.2. Let p be a prime number. Choose projections q_0, \dots, q_{p-1} in \mathcal{O}_∞^{st} with

- (1) $\sum_{j=0}^{p-1} q_j = 1$
- (2) $[q_j] = 0$ for $j = 1, \dots, p-1$ and $[q_0] = 1$ in $K_0(\mathcal{O}_\infty^{st}) \cong \mathbb{Z}$.

Set $u = \sum_{j=0}^{p-1} e^{2\pi i j/p} p_j$, which is a unitary of order p in \mathcal{O}_∞^{st} . We define an order p automorphism ν_p of \mathcal{O}_2 by

$$\nu_p = \bigotimes_{n=1}^{\infty} \text{Ad}(u).$$

Recall that the center of a simple unital C^* -algebra is trivial, this is, is equal to the set of scalar multiples of its unit. An action α of \mathbb{Z}_p on \mathcal{O}_2 is *pointwise outer* if α_g is not an inner automorphism for every nonzero $g \in \mathbb{Z}_p$.

Lemma 5.5.3. *Let p be a prime number, and denote by ν_p the order p automorphism of \mathcal{O}_2 constructed above. Then ν_p induces a pointwise outer action of \mathbb{Z}_p on \mathcal{O}_2 .*

Proof. We claim that it is enough to show that ν_p is not inner. Indeed, assume ν_p^j that is inner for some $j \in \{2, \dots, p-1\}$. Since $\{j, j^2, \dots, j^{p-1}\} = \mathbb{Z}_p$, it follows that ν_p is inner.

Assume that ν_p is inner, and let v be a unitary in \mathcal{O}_2 such that $\nu_p = \text{Ad}(v)$. For n in \mathbb{N} , set $A_n = \bigotimes_{k=1}^n \mathcal{O}_\infty^{st}$. Given $\varepsilon > 0$, there are a positive integer n in \mathbb{N} and a unitary w in A_n such that $\|v - w\| < \varepsilon$. Given a in \mathcal{O}_∞^{st} , denote by b its canonical image in the $(n+1)$ -st tensor factor of $\bigotimes_{k=1}^\infty \mathcal{O}_\infty^{st}$, this is, $b = 1 \otimes \dots \otimes 1 \otimes a \otimes 1 \dots$. Then

$$\begin{aligned} \|ua - au\| &= \|ua u^* - a\| \\ &= \|\nu(b) - b\| \\ &= \|v b v^* - b\| \\ &\leq 2\|v - w\| + \|w b w^* - b\|. \end{aligned}$$

Since w and b belong to different tensor factors, they commute, so we conclude that $\|ua - au\| < 2\varepsilon$. Since ε is arbitrary and a and u do not depend on it, it follows that $au = ua$. Since a is also arbitrary, it follows that u is in the center of \mathcal{O}_∞^{st} , which is trivial. This is a contradiction. Hence ν_p is not inner, and the result follows. \square

We compute the associated crossed product in the following lemma.

Lemma 5.5.4. *Let the notation be as in the previous lemma. Then the crossed product $\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_p$ is a unital Kirchberg algebra satisfying the UCT and with K -theory given by*

$$\left(K_0(\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_p), \left[1_{\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_p} \right], K_1(\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_p) \right) \cong \left(\mathbb{Z} \left[\frac{1}{p} \right], 0, \{0\} \right).$$

Proof. The crossed product is well known to be nuclear, unital and separable. Since ν_p is pointwise outer, [36, Theorem 3.1] implies that $\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_p$ is purely infinite and simple, and thus it is a Kirchberg algebra.

Given n in \mathbb{N} , set $A_n = \bigotimes_{k=1}^n \mathcal{O}_\infty^{st}$ and $u_n = \bigotimes_{k=1}^n u$. Note that $\mathcal{O}_2 = \varinjlim A_n$ and $\nu_p = \varinjlim \text{Ad}(u_n)$ with the obvious inductive limit decomposition and connecting maps. Moreover, there are natural isomorphism

$$\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_p \cong \varinjlim A_n \rtimes_{\text{Ad}(u_n)} \mathbb{Z}_p \cong A_n \oplus \dots \oplus A_n \cong \mathcal{O}_\infty^{st} \oplus \dots \oplus \mathcal{O}_\infty^{st}.$$

(There are p direct summands.) Since the UCT passes to inductive limits, we conclude that $\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_p$ satisfies the UCT.

The K -theory is computed analogously as it is done in the proof of [26, Lemma 4.7]. The crucial point is showing that the connecting maps in the inductive limit $\mathbb{Z}^p \rightarrow \mathbb{Z}^p \rightarrow \dots$ are all constant and given by the matrix

$$\begin{pmatrix} 1 & e^{2\pi i/p} & \dots & e^{2\pi i(p-1)/p} \\ e^{2\pi i/p} & e^{2\pi i2/p} & \dots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ e^{2\pi i(p-1)/p} & 1 & \dots & e^{2\pi i(p-2)/p} \end{pmatrix}.$$

We omit the details, our case is just notationally more difficult than the case $p = 2$. \square

We fix, until the end of this section, a prime number p . For a simple nuclear unital C^* -algebra A , denote by $\tilde{\alpha}_{A,p}$ the order p automorphism of $A \otimes \mathcal{O}_2$ defined by $\text{id}_A \otimes \nu_p$. By Kirchberg's \mathcal{O}_2 -isomorphism Theorem [35, Theorem 3.8], there is an isomorphism $\varphi: A \otimes \mathcal{O}_2 \rightarrow \mathcal{O}_2$. Denote by $\alpha_{A,p}$ the order p automorphism of \mathcal{O}_2 given by

$$\alpha_{A,p} = \varphi \circ \tilde{\alpha}_{A,p} \circ \varphi^{-1}.$$

It is immediate to check that $\alpha_{A,p}$ is approximately representable, using that ν_p is approximately representable.

We will denote by D_p the tensor product $\mathcal{O}_\infty^{st} \otimes \mathbb{M}_{p^\infty}$. Suppose that α and β are actions of a discrete group G on C^* -algebras A and B . By definition α and β are *conjugate* if there is an isomorphism γ from A to B such that $\beta_g \circ \gamma = \gamma \circ \alpha_g$ for every $g \in G$. Similarly α and β are *cocycle conjugate* if β is conjugate to a cocycle perturbation of α .

Proposition 5.5.5. *Let A and B be D_p -absorbing unital Kirchberg algebras. Then the following statements are equivalent:*

- (1) *A and B are isomorphic;*
- (2) *The actions $\alpha_{A,p}$ and $\alpha_{B,p}$ are conjugate;*
- (3) *The actions $\alpha_{A,p}$ and $\alpha_{B,p}$ are cocycle conjugate;*
- (4) *The crossed products $\mathcal{O}_2 \rtimes_{\alpha_{A,p}} \mathbb{Z}_p$ and $\mathcal{O}_2 \rtimes_{\alpha_{B,p}} \mathbb{Z}_p$ are isomorphic.*

Proof. (1) implies (2). If $\psi: A \rightarrow B$ is an isomorphism, then $\psi \otimes \text{id}_{\mathcal{O}_2}: A \otimes \mathcal{O}_2 \rightarrow B \otimes \mathcal{O}_2$ conjugates $\text{id}_A \otimes \nu_p$ and $\text{id}_B \otimes \nu_p$, and hence $\alpha_{A,p}$ and $\alpha_{B,p}$ are conjugate.

(2) implies (3) and (3) implies (4) hold in full generality.

(4) implies (1). There are isomorphisms

$$\begin{aligned} \mathcal{O}_2 \rtimes_{\alpha_{A,p}} \mathbb{Z}_p &\cong (A \otimes \mathcal{O}_2) \rtimes_{\text{id}_A \otimes \nu_p} \mathbb{Z}_p \\ &\cong A \otimes (\mathcal{O}_2 \rtimes_{\nu_p} \mathbb{Z}_p) \\ &\cong A \otimes D_p. \end{aligned}$$

Likewise, $\mathcal{O}_2 \rtimes_{\alpha_{B,p}} \mathbb{Z}_p \cong B \otimes D_p$. The result now follows from the fact that A and B are D_p -absorbing. \square

5.6. Borel reduction. Denote by \mathcal{A}_p the set of all elements γ in $\Gamma(H)$ such that $C^*(\gamma)$ is a unital D_p -absorbing Kirchberg algebra. Then \mathcal{A}_p is Borel by Theorem 5.1.3 and Corollary 5.3.5. One can, in fact, regard \mathcal{A}_p as the standard Borel space parametrizing D_p -absorbing unital Kirchberg algebras. Thus, the equivalence relation E on \mathcal{A}_p defined by

$$\gamma E \gamma' \quad \text{if and only if} \quad C^*(\gamma) \cong C^*(\gamma'),$$

can be identified with the relation of isomorphism of unital D_p -absorbing Kirchberg algebras.

Theorem 5.6.1. *There are Borel reductions:*

- (1) *From the relation of isomorphism of D_p -absorbing Kirchberg algebras, to the relation of cocycle conjugacy of approximately representable automorphisms of \mathcal{O}_2 of order p .*
- (2) *From the relation of isomorphism of D_p -absorbing Kirchberg algebras, to the relation of conjugacy of approximately representable automorphisms of \mathcal{O}_2 of order p .*

Proof. In view of Proposition 5.5.5, and Elliott's theorem $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ [48], it is enough to show that there is a Borel function from $\Gamma_{uKir}(H)$ to $\text{Aut}_p(\mathcal{O}_2 \otimes \mathcal{O}_2)$ that assigns to every $\gamma \in \Gamma_{uKir}(H)$, an automorphism $\alpha_{\gamma,p}$ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ which is conjugate to $\text{id}_{C^*(\gamma)} \otimes \nu_p$.

We follow the notation of [19, Section 6.1], and denote by $SA(\mathcal{O}_2)$ the space of C^* -subalgebras of \mathcal{O}_2 . Then $SA(\mathcal{O}_2)$ is a Borel subset of the Effros Borel space of closed subsets of \mathcal{O}_2 , as defined in [31, Section 12.C]. It follows from [19, Theorem 6.5] that the set $SA_{uKir}(\mathcal{O}_2)$ of C^* -subalgebras of \mathcal{O}_2 isomorphic to a unital Kirchberg algebra is Borel. Moreover, again by [19, Theorem 6.5], there is a Borel function from $\Gamma_{uKir}(\mathcal{O}_2)$ to $SA_{uKir}(\mathcal{O}_2)$ that assigns to an element γ of $\Gamma_{uKir}(\mathcal{O}_2)$ a subalgebra of \mathcal{O}_2 isomorphic to $C^*(\gamma)$. It is therefore enough to show that there is a Borel function from $SA_{uKir}(\mathcal{O}_2)$ to $\text{Aut}_p(\mathcal{O}_2 \otimes \mathcal{O}_2)$ that assigns to $A \in SA_{uKir}(\mathcal{O}_2)$ an automorphism $\alpha_{A,p}$ of $\mathcal{O}_2 \otimes \mathcal{O}_2$ conjugate to $\text{id}_A \otimes \nu_p$.

Denote by $\text{End}(\mathcal{O}_2 \otimes \mathcal{O}_2)$ the space of endomorphism of $\mathcal{O}_2 \otimes \mathcal{O}_2$. By [19, Theorem 7.6], there is a Borel map from $SA_{uKir}(\mathcal{O}_2)$ to $\text{End}(\mathcal{O}_2 \otimes \mathcal{O}_2)$ that assigns to an element A in $SA_{uKir}(\mathcal{O}_2)$ a unital injective endomorphism η_A of $\mathcal{O}_2 \otimes \mathcal{O}_2$ with range $A \otimes \mathcal{O}_2$. In particular, η_A is an isomorphism between $\mathcal{O}_2 \otimes \mathcal{O}_2$ and $A \otimes \mathcal{O}_2$. For A in $SA_{uKir}(\mathcal{O}_2)$, define

$$\alpha_{A,p} = \eta_A^{-1} \circ (\text{id}_A \otimes \nu_p) \circ \eta_A,$$

and note that the map $A \mapsto \alpha_{A,p}$ is Borel.

It is enough to show that for every $x, y \in \mathcal{O}_2$ and every $\varepsilon > 0$, the set of all C^* -algebras A in $SA_{uKir}(\mathcal{O}_2)$ such that

$$\|\alpha_{A,p}(x) - y\| < \varepsilon,$$

is Borel. Fix x and y in \mathcal{O}_2 .

By [31, Theorem 12.13], there is a sequence of Borel functions from $SA_{uKir}(\mathcal{O}_2)$ to \mathcal{O}_2 , which we will denote by $A \mapsto a_n^A$ for n in ω , such that for A in $SA_{uKir}(\mathcal{O}_2)$, the set $\{a_n^A : n \in \omega\}$ is an enumeration of a dense subset of A .

Fix a countable dense subset $\{b_n : n \in \omega\}$ of \mathcal{O}_2 . Then

$$\|\alpha_{A,p}(x) - y\| = \|(\text{id}_A \otimes \nu)(\eta_A(x)) - \eta_A(y)\|,$$

and thus $\|\alpha_{A,p}(x) - y\| < \varepsilon$ if and only if there are positive integers $k \in \omega$ and $n_0, \dots, n_{k-1}, m_0, \dots, m_{k-1} \in \omega$, and scalars $\lambda_0, \dots, \lambda_{k-1} \in \mathbb{Q}(i)$, such

that

$$\left\| \eta_A(x) - \sum_{i \in k} \lambda_i a_{n_i}^G \otimes b_{m_i} \right\| < \frac{\varepsilon}{2} \quad \text{and} \quad \left\| \sum_{i \in k} \lambda_i a_{n_i}^G \otimes \nu(b_{m_i}) - \eta_A(y) \right\| < \frac{\varepsilon}{2}.$$

Since the map $A \mapsto \eta_A$ is Borel, it follows that the set of all C^* -algebras A in $SA_{uKir}(\mathcal{O}_2)$ such that $\|\alpha_{A,p}(x) - y\| < \varepsilon$ is Borel. The result follows. \square

5.7. Constructing Kirchberg algebras with assigned K_0 -group. The following is the main result of this section.

Theorem 5.7.1. *There is a Borel map from the Borel space \mathcal{G} of discrete groups to the Borel space $\Gamma_{uKir}(H)$ parametrizing Kirchberg algebras, which assigns to every countable discrete abelian group G , a code γ for a unital Kirchberg algebra $C^*(\gamma)$ that satisfies the UCT, and with K -theory given by*

$$(K_0(C^*(\gamma)), [1_{C^*(\gamma)}], K_1(C^*(\gamma))) \cong (G, 0, \{0\}).$$

In particular, $C^(\gamma)$ is \mathcal{O}_∞^{st} -absorbing. Moreover, if p is a prime number, then $C^*(\gamma)$ is D_p -absorbing if and only if G is uniquely p -divisible.*

Proof. Use Lemma 2.11.1 to choose, in a Borel way from G , a torsion free abelian group H and an automorphism α of H such that

$$H/\text{Im}(\text{id}_H - \alpha) \cong G.$$

Denote by D the dimension group given by

$$D = \mathbb{Z} \left[\frac{1}{2} \right] \oplus H$$

with positive cone

$$D^+ = \{(t, h) \in D : t > 0\} \cup \{(0, 0)\},$$

and order unit $(1, 0)$. Consider the endomorphism ρ of D defined by

$$\beta(t, h) = \left(\frac{t}{2}, \alpha(h) \right)$$

for (t, h) in D . It is clear that D and β can be computed in a Borel way from H and α . By Corollary 4.5.2, one can obtain in a Borel way from H and β , a code for a unital AF-algebra B and a code for an injective corner endomorphism ρ of B such that the K_0 -group of B is isomorphic to D , and the endomorphism of the K_0 -group of B induced by ρ is conjugate to β . By Corollary 3.5.4, one can obtain in a Borel way a code $\gamma_G \in \Gamma(H)$ for the crossed product $B \rtimes_\rho \mathbb{N}$ of B by the endomorphism ρ . It can be shown, as in the proof of [49, Theorem 3.6], that $C^*(\gamma_G)$ is a unital Kirchberg algebra satisfying the UCT, with trivial K_1 -group, K_0 -group isomorphic to G , and $[1_{C^*(\gamma_G)}] = 0$ in $K_0(C^*(\gamma))$. An easy application of the Pimsner-Voiculescu exact sequence [47, Theorem 2.4] gives the computation of the K -theory; see [49, Corollary 2.2]. Pure infiniteness of $C^*(\gamma_G)$ is proved in [49, Theorem 3.1]. The map $\mathcal{G} \rightarrow \Gamma_{uKir}(H)$ given by $G \mapsto \gamma_G$ is Borel by construction. \square

Corollary 5.7.2. *Let p be a prime number. There are Borel reductions:*

- (1) *From the relation of isomorphism of infinite countable discrete abelian groups, to the relation of isomorphism of $\mathcal{O}_\infty^{\text{st}}$ -absorbing unital Kirchberg algebras (satisfying the UCT and with trivial K_1 -group).*
- (2) *From the relation of isomorphism of uniquely p -divisible infinite countable discrete abelian groups, to the relation of isomorphism of D_p -absorbing unital Kirchberg algebras (satisfying the UCT and with trivial K_1 -group).*

Proof. Both results follow from Theorem 5.7.1 above, together with the Kirchberg-Phillips classification theorem (see [34] and [45]). \square

Corollary 5.7.3. *Let p be a prime number, and \mathcal{C} be any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of elements of \mathcal{C} is Borel. Assume that F be any of the following equivalence relations:*

- *isomorphism of D_p -absorbing unital Kirchberg algebras with trivial K_1 -group,*
- *conjugacy of approximately representable automorphisms of \mathcal{O}_2 of order p ,*
- *cocycle conjugacy of approximately representable automorphisms of \mathcal{O}_2 of order p .*

Then $\cong_{\mathcal{C}}$ is Borel reducible to F , and moreover F is a complete analytic set.

Corollary 5.7.4. *The relations of isomorphism of unital Kirchberg algebras, and conjugacy and cocycle conjugacy of automorphisms of \mathcal{O}_2 , are complete analytic sets, and in particular, not Borel.*

5.8. Actions of countable discrete groups on \mathcal{O}_2 . Suppose that G is a countable discrete group. Denote by $\text{Act}(G, A)$ the space of actions of G on A endowed with the topology of pointwise norm convergence. It is clear that $\text{Act}(G, A)$ is homeomorphic to a G_δ subspace of the product of countably many copies of A and, in particular, is a Polish space.

Let G and H be countable discrete groups, and let $\pi: G \rightarrow H$ be a surjective homomorphism from G to H . Define the Borel map $\pi^*: \text{Act}(H, A) \rightarrow \text{Act}(G, A)$ by $\pi^*(\alpha) = \alpha \circ \pi$ for α in $\text{Act}(H, A)$. It is easy to check that π^* is a Borel reduction from the relation of conjugacy of actions of H to the relation of conjugacy of actions of G . The following proposition is then an immediate consequence of this observation together with Corollary 5.7.3.

Proposition 5.8.1. *Let G be a countable discrete group with a nontrivial cyclic quotient. If \mathcal{C} is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of elements of \mathcal{C} is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation of conjugacy of actions of G on \mathcal{O}_2 .*

Moreover, the latter equivalence relation is a complete analytic set as a subset of $\text{Act}(G, A) \times \text{Act}(G, A)$ and, in particular, is not Borel.

The situation for cocycle conjugacy is not as clear. It is not hard to verify that if $G = H \times N$ and $\pi: G \rightarrow H$ is the canonical projection, then π^* , as defined before, is a Borel reduction from the relation of cocycle conjugacy in $\text{Act}(H, A)$ to the relation of cocycle conjugacy in $\text{Act}(G, A)$. Using this observation and the structure theorem for finitely generated abelian groups, one obtains as a consequence of Corollary 5.7.3 the following fact:

Proposition 5.8.2. *Let G be any finitely generated abelian group. If \mathcal{C} is any class of countable structures such that the relation $\cong_{\mathcal{C}}$ of isomorphism of elements of \mathcal{C} is Borel, then $\cong_{\mathcal{C}}$ is Borel reducible to the relation of conjugacy of actions of G on \mathcal{O}_2 .*

Moreover, the latter equivalence relation is a complete analytic set as a subset of $\text{Act}(G, A) \times \text{Act}(G, A)$ and, in particular, not Borel.

6. FINAL COMMENTS AND REMARKS

Recall that an automorphism of a C^* -algebra A is said to be *pointwise outer* (or *aperiodic*) if none of its nonzero powers is inner. By [41, Theorem 1], an automorphism of a Kirchberg algebra is pointwise outer if and only if it has the so called Rokhlin property. Moreover, it follows from this fact together with [46, Corollary 5.14] that the set $\text{Rok}(A)$ of pointwise outer automorphisms of a Kirchberg algebra A is a dense G_δ subset of $\text{Aut}(A)$, which is moreover easily seen to be invariant by cocycle conjugacy.

It is an immediate consequence of [41, Theorem 9], see also [39, Theorem 5.2], that aperiodic automorphisms of \mathcal{O}_2 form a single cocycle conjugacy class. In particular, and despite the fact that the relation of cocycle conjugacy of automorphisms of \mathcal{O}_2 is not Borel, its restriction to the comeager subset $\text{Rok}(\mathcal{O}_2)$ of $\text{Aut}(\mathcal{O}_2)$ has only one class and, in particular, is Borel. This can be compared with the analogous situation for the group of ergodic measure preserving transformations of the Lebesgue space: The main result of [21] asserts that the relation of conjugacy of ergodic measure preserving transformations of the Lebesgue space is a complete analytic set. On the other hand, the restriction of such relation to the comeager set of ergodic *rank one* measure preserving transformations is Borel.

It is conceivable that similar conclusions might hold for the relation of conjugacy of automorphisms of \mathcal{O}_2 . We therefore suggest the following problem:

Question 6.1. *Consider the relation of conjugacy of automorphisms of \mathcal{O}_2 , and restrict it to the invariant dense G_δ set of aperiodic automorphisms. Is this equivalence relation Borel?*

By [33, Theorem 4.5], the automorphisms of \mathcal{O}_2 are not classifiable up to conjugacy by countable structures. This means that if \mathcal{C} is any class of countable structures, then the relation of conjugacy of automorphisms of \mathcal{O}_2 is *not* Borel reducible to the relation of isomorphisms of structures from \mathcal{C} . It would be interesting to know if one can draw similar conclusions for the relation of cocycle conjugacy.

Question 6.2. *Is the relation of cocycle conjugacy of automorphisms of \mathcal{O}_2 classifiable by countable structures?*

[33, Theorem 4.5] in fact shows that the relation of conjugacy of automorphisms is not classifiable for a large class of C^* -algebras, including all C^* -algebras that are classifiable according to the Elliott classification program [51, Section 2.2]. It would be interesting to know if the same holds for the relation of cocycle conjugacy. More generally, it would be interesting to have some information about the complexity of the relation of cocycle conjugacy for automorphisms of other simple C^* -algebras. This problem seems to be currently wide open.

Problem 6.3. Find an example of a simple unital nuclear separable C^* -algebra for which the relation of cocycle conjugacy of automorphisms is not classifiable by countable structures.

Recall that an equivalence relation on a standard Borel space is said to be *smooth*, or *concretely classifiable*, if it is Borel reducible to the relation of equality in some standard Borel space. A smooth equivalence relation is in particular Borel and classifiable by countable structures. Therefore Corollary 5.7.4 in particular shows that the relation of cocycle conjugacy of automorphisms of \mathcal{O}_2 is not smooth.

If X is a compact Hausdorff space, we denote by $C(X)$ the unital commutative C^* -algebra of complex-valued continuous functions on X . It is a classical result of Gelfand and Naimark that any unital commutative C^* -algebra is of this form; see [1, Theorem II.2.2.4]. Moreover, by [1, II.2.2.5], the group $\text{Aut}(C(X))$ of automorphisms of $C(X)$ is isomorphic to the group $\text{Homeo}(X)$ of homeomorphisms of X . It is clear that in this case the relations of conjugacy and cocycle conjugacy of automorphisms coincide. By [4, Theorem 5], if X is the Cantor set, then the relation of (cocycle) conjugacy of automorphisms of $C(X)$ is not smooth (but classifiable by countable structures). On the other hand, when X is the unit square $[0, 1]^2$, then the relation of cocycle conjugacy of automorphisms of $C(X)$ is not classifiable by countable structures in view of [24, Theorem 4.17]. This addresses Problem 6.3 in the case of abelian unital C^* -algebras. No similar examples are currently known for simple unital C^* -algebras.

It is worth mentioning here that if one considers instead the relation of *unitary conjugacy* of automorphisms, then there is a *strong dichotomy* in the complexity. Recall that two automorphisms α, β of a unital C^* -algebra are unitarily conjugate if $\alpha \circ \beta^{-1}$ is an inner automorphism, this is, implemented by a unitary element of A . Theorem 1.2 in [37] shows that whenever this relation is not smooth, then it is even not classifiable by countable structures. The same phenomenon is shown to hold for unitary conjugacy of *irreducible representations* in [32, Theorem 2.8.]; see also [44, Section 6.8]. It is possible that similar conclusions might hold for the relation of conjugacy or cocycle conjugacy of automorphisms of simple C^* -algebras.

Question 6.4. *Is it true that, whenever the relation of (cocycle) conjugacy of automorphisms of a simple unital C^* -algebra A is not smooth, then it is not even classifiable by countable structures?*

Kirchberg-Phillips classification theorem (Theorem 5.2.3) asserts that Kirchberg algebras satisfying the UCT are classified up to isomorphism by their K -groups. By [18, Section 3.3] the K -theory of a C^* -algebra can be computed in a Borel way. It follows that Kirchberg algebras satisfying the UCT are classifiable up to isomorphism by countable structures. Conversely, by Corollary 5.7.3 if \mathcal{C} is any class of countable structure *with Borel isomorphism relation*, then the relation of isomorphism of elements of \mathcal{C} is Borel reducible to the relation of isomorphism of Kirchberg algebras satisfying the UCT. It is natural to ask whether the same conclusion holds for *any* class of countable structures \mathcal{C} .

Question 6.5. *Suppose that \mathcal{C} is a class of countable structures. Is the relation of isomorphism of elements of \mathcal{C} Borel reducible to the relation of isomorphism of Kirchberg algebras with the UCT?*

A class \mathcal{D} of countable structures is *Borel complete* if the following holds: For any class of countable structures \mathcal{C} the relation of isomorphism of elements of \mathcal{C} is Borel reducible to the relation of isomorphism of elements of \mathcal{D} . Theorem 1, Theorem 3, and Theorem 10 of [22] assert that the classes of countable trees, countable linear orders, and countable fields of any fixed characteristic are Borel complete; Theorem 7 of [22] shows, using results of Mekler from [40], that the relation of isomorphism of countable groups is Borel complete. A long standing open problem –first suggested in [22]– asks whether the class of (torsion-free) abelian groups is Borel complete. In view of Corollary 5.7.2, a positive answer to such problem would settle Question 6.5 affirmatively.

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